Quadratic Approximation

Back to

$$f(x) = f(b) + f'(b)(x-b) + \int_b^x f''(t)(x-t)\,dt$$

integration by parts

$u = f''(t) \qquad dv = (x-t)\,dt$

$du = f'''(t) \qquad v = -\frac{(x-t)^2}{2}$

$$f(x) = f(b) + f'(b)(x-b) + f''(t)\left(-\frac{(x-t)^2}{2}\right)\Bigg|_b^x + \int_b^x f'''(t)\frac{(x-t)^2}{2}\,dt$$

$$\underset{\text{Actual}}{f(x)} = \underbrace{f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2}_{\substack{Q(x) \text{ or } T_2(x) \\ \text{Approximate}}} + \underbrace{\int_b^x f'''(t)\frac{(x-t)^2}{2}\,dt}_{\text{error}}$$

ex: $f(x) = e^x \qquad b = 0$

$f''(x) = e^x \quad f''(0) = 1$

$$T_2(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 = 1 + x + \frac{x^2}{2}$$

$$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2} = 1.1 + \frac{0.01}{2}$$
$$= 1.105$$

For the error bound:

Let M be such that

$|f'''(t)| \leq M$ for t between b & x

$$\boxed{|f(x) - T_2(x)| \leq \frac{M\,|x-b|^3}{2 \cdot 3}}$$

in my example with $e^{0.1} \approx 1.105$ above

$f'''(t) = e^t$ - take $M \geq e^{0.1}$

$\cancel{|e^x - (1 + x + \frac{x^2}{2})| \leq e}$

$e \leq 2^{10}$

$$|e^{0.1} - (1.105)| \leq \frac{e^{0.1} \cdot |0.1|^3}{6} \leq \frac{2(0.001)}{6}$$
$$\approx 0.0003$$

example: $f(x) = (1+x)^{2/3}$

(i) Find the quadratic approximation $T_2(x)$ near 7 ← b is always "nice" $f(b), f'(b), \ldots$ "easy"

(ii) Find an upper bound for the error if x is in $[6,8]$. ← common to have interval centered at b.

(iii) Approximate $7.5^{2/3}$.

(iv) Compare error from calculator with error upper bound.

(i) $T_2(x) = f(7) + f'(7)(x-7) + \frac{f''(7)}{2}(x-7)^2$

$f(x) = (1+x)^{2/3}$ $\quad f(7) = 8^{2/3} = 4$

$f'(x) = \frac{2}{3}(1+x)^{-1/3}$ $\quad f'(7) = \frac{2}{3} 8^{-1/3} = \frac{1}{3}$

$f''(x) = \frac{-2}{9}(1+x)^{-4/3}$ $\quad f''(7) = \frac{-2}{9} 8^{-4/3} = \frac{-1}{72}$

$T_2(x) = 4 + \frac{1}{3}(x-7) - \frac{1}{144}(x-7)^2$

(ii) $f'''(t) = \frac{8}{27}(1+t)^{-7/3}$

$6 \le t \le 8$ $\qquad \left| \frac{8}{27(1+t)^{7/3}} \right| \le \frac{8}{27(1+6)^{7/3}} = M$

Note: where does t come from?

(a) We may have a single x.
On Monday I had $e^{0.1}$, $x = 0.1$
$b = 0$, $0 \le t \le 0.1$

More common

(b) x is in an interval
then t comes from the same interval.

To make $\frac{1}{A}$ as large as possible

you make A as small as possible

$$|f(x) - T_2(x)| \le \frac{8}{27(1+6)^{7/3}} \frac{|x-7|^3}{6}$$

$$\le \underbrace{\frac{8}{27 \cdot 7^{7/3}} \frac{1^3}{6}}_{\text{error upper bound}}$$

$6 \le x \le 8$
$|x-7| \le 1.$

error upper bound (≈ 0.0005) check

$f(x) = (1+x)^{2/3}$

(~~iv~~ iii) $7.5^{2/3} = (1+6.5)^{2/3} = f(6.5)$

$$\approx T_2(6.5) = 4 + \frac{1}{3}(6.5-7) - \frac{1}{144}(6.5-7)^2$$

$$\approx 3.8316$$

(iv) Compare error bound in (ii) with $|7.5^{2/3} - T_2(6.5)|$ in your calculator.

should be much less