Taylor Polynomials + Taylor Series

Linear Approximation (you know this)

$y=f(x)$ point of ~~on~~ tangency at $x=b$

function value $f(b)$

slope $f'(b)$

Tangent Line: $y-f(b)=f'(b)(x-b)$

$$y=\underbrace{f(b)+f'(b)(x-b)}_{L(x)}$$

$T_1(x)$ - degree 1 Taylor approximatio

ex: $f(x)=e^x$, $b=0$ $\qquad f(0)=1$

$f'(x)=e^x \qquad f'(0)=1$

$$T_1(x)=1+1(x-0)=1+x$$

$e^x \approx 1+x$ when x is near 0

So, $e^{0.01} \approx 1.01$ OK

$e^{0.1} \approx 1.1$

$e^{0.5} \approx 1.5$?

$2.7 \approx e^{1} \approx 1+1=2$ NOT

how close is close enough to 0?

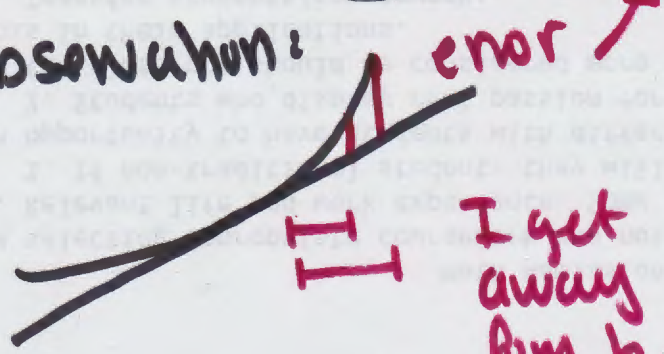
$$|\text{Error}| = |\text{Actual Value} - \text{Approx. Value}|$$

we won't know

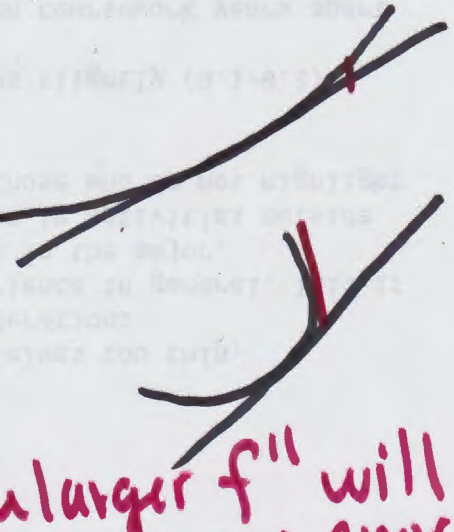
we compute with $T_1(x)$

We'll compute an <u>upper bound</u> for $|\text{error}| = |f(x) - T_1(x)|$.

Observation:



I get away from b.



The larger f'' will cause larger error.

Start with Fundamental Theorem of Calculus:

$$f(x) - f(b) = \int_b^x f'(t)\,dt$$

(b + x are like constant)

Do integration by parts

$u = f'(t)$ $\quad$ $dv = dt$

$du = f''(t)\,dt$ $\quad$ $v = t - x$

$$f(x) - f(b) = f'(t)(t-x)\Big|_b^x - \int_b^x f''(t)(t-x)\,dt$$

$$f(x) - f(b) = -f'(b)(b-x) + \int_b^x f''(t)\,(x-t)\,dt$$

$$\underset{\uparrow \atop \text{actual}}{f(x)} = \underbrace{f(b) + f'(b)(x-b)}_{\substack{T_1(x) \\ \text{approximate}}} + \underbrace{\int_b^x f''(t)(x-t)\,dt}_{\text{ERROR}}$$

We want to ~~BOUND~~ the error

If $b < x$:

$$\left| \int_b^x f''(t)\,(x-t)\,dt \right| \leq \int_b^x \boxed{|f''(t)|}\,(x-t)\,dt$$

Let M be a (positive) number such that
$$|f''(t)| \leq M$$
when t is between b and x.

$$\leq \int_b^x M\,(x-t)\,dt$$

$$= M \frac{(x-t)^2}{2}\Big|_b^x = \frac{M(x-b)^2}{2}$$

If $b > x$, work with integrals $\int_x^b \ldots$

in any case

$$|f(x) - T_1(x)| \le \frac{M\,|x-b|^2}{2}$$ Taylor's Inequality

where M is such that $|f''(t)| \le M$
when t is between b & x.

Back to $e^{0.01} \approx 1.01$ $\quad b = 0$, $x = 0.01$

$f(x) = e^x$
$f'(x) = e^x$
$f''(t) = e^t$

$|e^t| = e^t$ when t is between 0 and 0.01

increasing function

$M \geq$ maximum value of e^t on this interval

take $\boxed{M = e^{0.01}}$ OR M any number $M \geq e^{0.01}$

$\boxed{M = 2}$

note: we usually take M to be the maximum of $|f''(t)|$ in the given interval.

$$|e^x - (1+x)| \leq \frac{2|x-0|^2}{2}$$

In this problem, I know $x = 0.01$

$$|e^{0.01} - 1.01| \leq \frac{2|0.01|^2}{2} = 0.0001$$

example: Approximate $f(x) = \sin x$ on the interval $[-0.2, 0.2]$. Find $T_1(x)$ & error upper bound.

old ① $T_1(x)$

$f(x) = \sin x$ $\quad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\quad$ $f'(0) = \cos 0 = 1$

$$T_1(x) = 0 + 1(x-0) = x$$

$\sin x \approx x$ when x is near 0.

new! (2) Bounding error $|\sin x - x| \leq ?$

$f''(t) = -\sin t$

$|-\sin t| \leq 1$

note: with $\sin x$ + $\cos x$ we always use $M=1$.

$$\underset{\substack{\uparrow \\ f(x)}}{|\sin x} - \underset{\substack{\uparrow \\ T_1(x)}}{x|} \leq \frac{1 \cdot |x-0|^2}{2} \leq \frac{(0.2)^2}{2} = 0.02$$

note: when we don't know x, but we have an interval x comes from, we bound $|x-b|$.

Most common: interval centered at b.

$|x-b| \leq$ half the length of the interval.

[b]

$|\sin x - x| \leq 0.02$ when x is in $[-0.2, 0.2]$.