15.4 Application (really just center of mass) of Double Integrals

We already have

① $f(x,y) \geq 0$, $\iint_D f(x,y)\,dA$ is <u>volume</u> under $z = f(x,y)$ over region D on xy-plane

② $f(x,y) > g(x,y)$ (on D)

$\iint_D f(x,y) - g(x,y)\,dA$: volume between $z = f(x,y)$ and $z = g(x,y)$

③ $\iint_D dA$ = Area of D

"Sum pieces to get the whole"

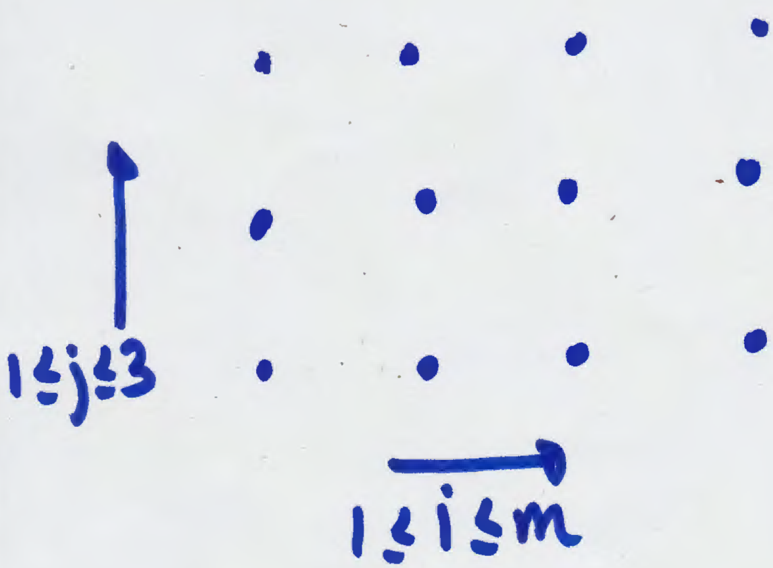
$$\text{Average value of } f(x,y) \text{ over } D = \frac{1}{\text{area } D} \iint_D f(x,y)\,dA$$

one variable version:

$$\text{Average value of } f(x) \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx$$

brief explanation with example:

Averaging 12 things

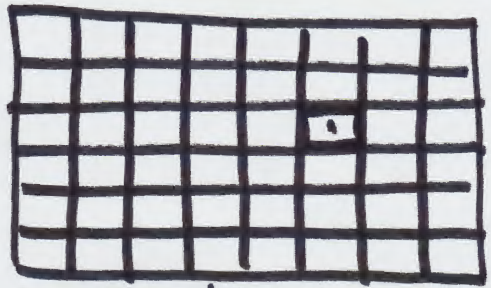


$$\frac{1}{2\cdot 3}\sum_{i=1}^{4}\sum_{j=1}^{3} n(i,j)$$

continuous example:

plate temperature $f(x,y)$

$1 \le j \le n$



$1 \le i \le m$

How do I find average temperature?

like I'm averaging nm values.

$$\frac{1}{\underbrace{\boxed{\Delta A\, nm}}_{\text{Area}}} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_i^*)\, \Delta A = \frac{1}{\text{Area}} \underbrace{\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_i^*)\, \Delta A}_{\text{take } n,m \to \infty}$$

$$\boxed{\text{Average Value} = \frac{1}{\text{Area}} \iint_R f(x,y)\, dA}$$

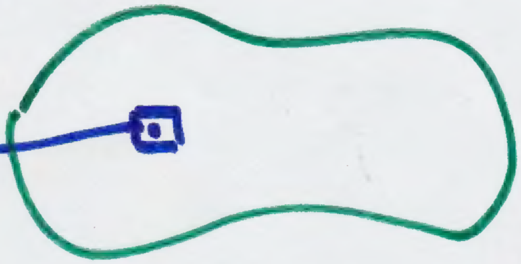
It works for non-rectangular regions, too.

CENTER OF MASS

Problem:

2D-plate
"lamina"

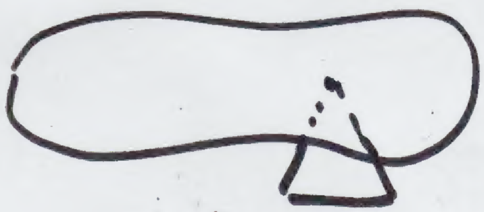
$\rho(x,y) \approx \dfrac{\Delta \text{mass}}{\Delta \text{area}}$

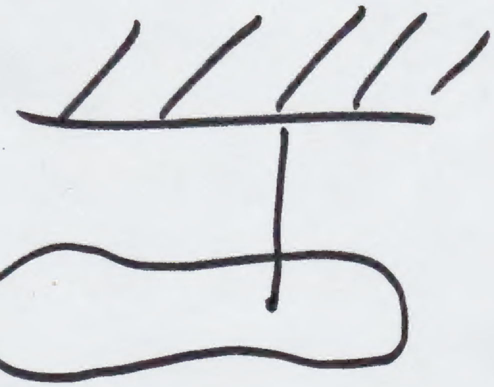


with density function

mass / unit area (kg) or weight / unit area (lb)

We'll be given $\rho(x,y)$.

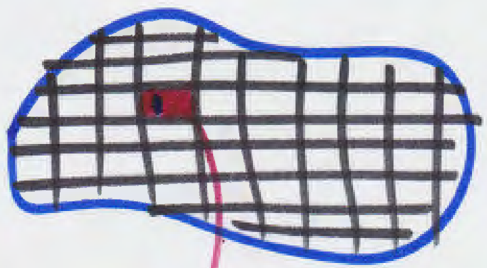




Computing (mass) of lamina from $\rho(x,y)$

Any integration problem has 3 steps

① Take the whole + chop it up



② Estimate each ~~peace~~ piece

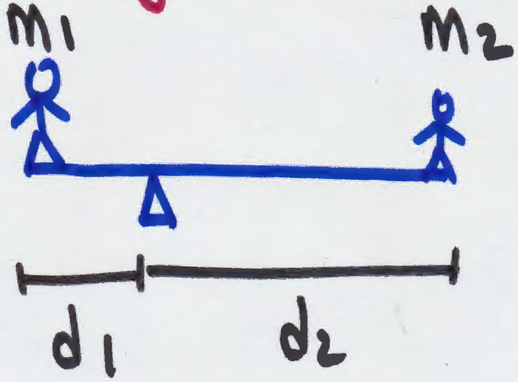
$$\Delta \text{mass} \approx \rho(x,y)\, \Delta \text{Area}$$

③ $\int$um the pieces to get the whole

$$\boxed{\text{Total mass} = \iint_D \rho(x,y)\, dA}$$

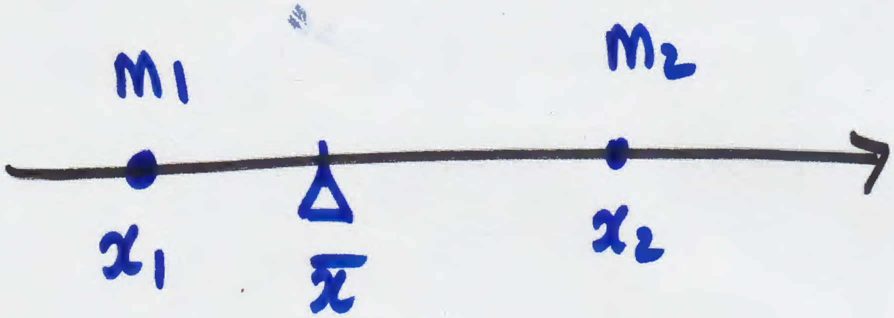
Computing center of mass

(1)



$$m_1 d_1 = m_2 d_2$$

(2)



$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

clean up $\bar{x} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

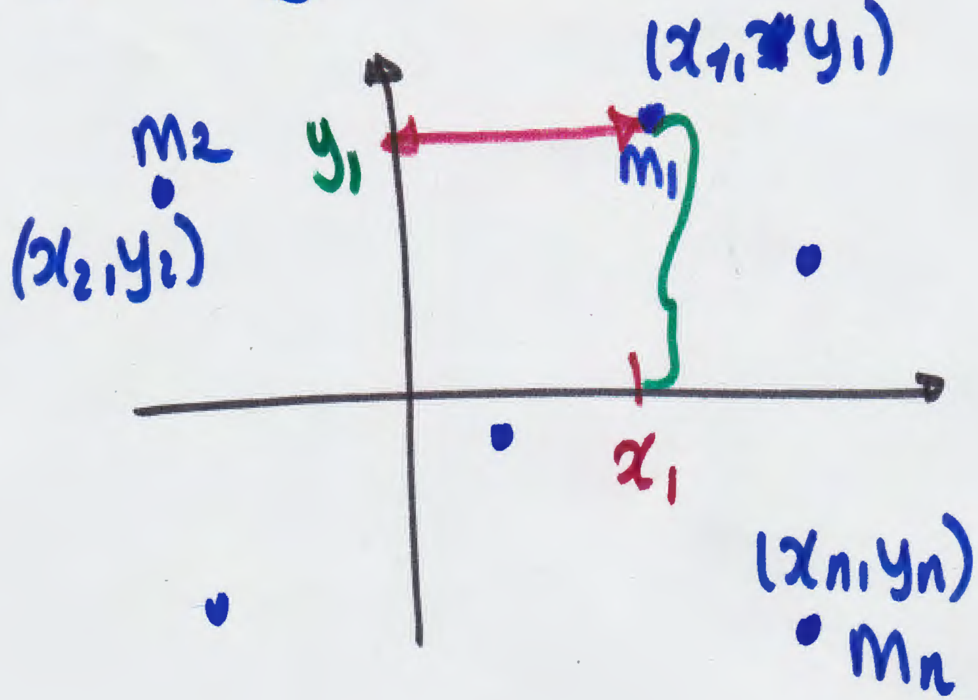
moment

total mass

Generalize to $m_1, \dots, m_2$ at $x_1, \dots, x_n$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + \dots + m_n}$$

③ xy-plane



$$M_y = m_1 x_1 + \cdots + m_n x_n$$ moment about y-axis

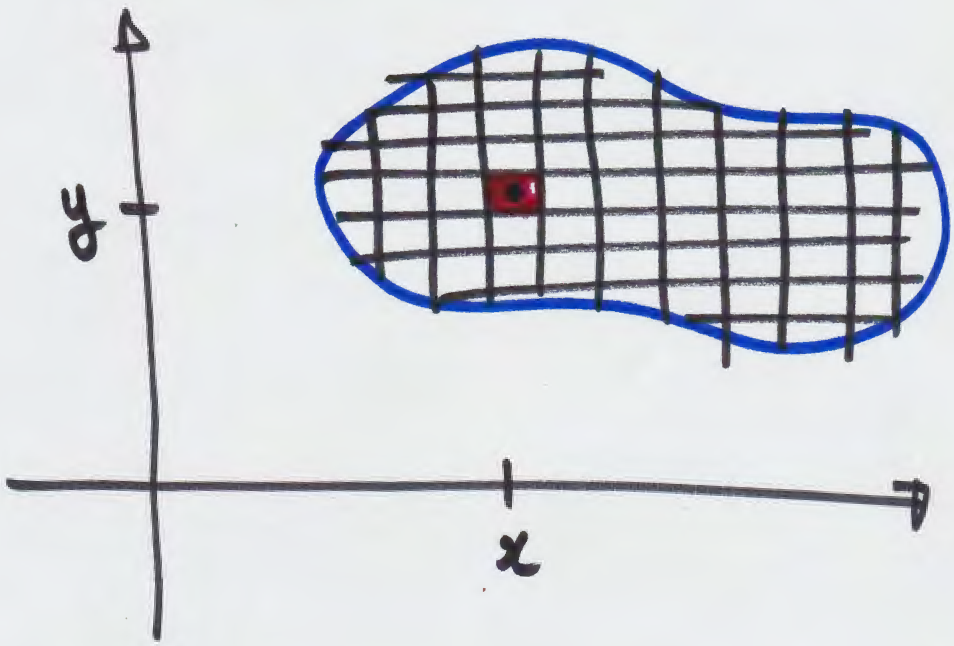
x-coordinate gives distance to y-axis

$$M_x = m_1 y_1 + \cdots + m_n y_n$$ moment about x-axis

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

$$M = m_1 + \cdots + m_n$$ total mass

④ Going from discreet to continuous



density function
$\rho(x,y)$
"ro"

~~Three~~ Three steps of integration

① Chop it up.

② Estimate pieces (moments)

$\Delta \text{mass} \cdot x$ OR $\Delta \text{mass} \cdot y$

$\approx \rho(x,y) \cdot \Delta \text{Area} \cdot x$ OR $\approx \rho(x,y) \Delta \text{Area} \cdot y$

③ Sum the pieces

$$\boxed{M_y = \iint_D x \rho(x,y)\, dA}$$

$$\boxed{M_x = \iint_D y \rho(x,y)\, dA}$$

$\bar{x} = \dfrac{M_y}{M}$, $\bar{y} = \dfrac{M_x}{M}$

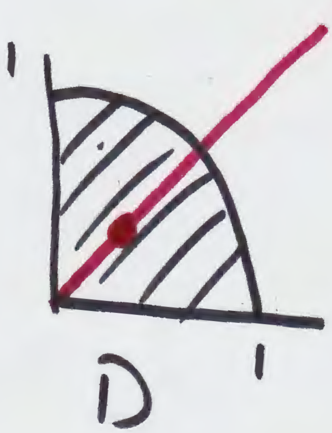
example: Find the center of mass of the lamina that occupies part of the disk $x^2+y^2 \le 1$ in the first quadrant if the density at any point is proportional to the square of its distance from the origin.

$$\rho(x,y) \sim \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$$

so $\boxed{\rho(x,y) = k(x^2+y^2)}$, $k>0$

density

switch $x + y$
→ same function



BOTH $\rho(x,y)$ **and** D have symmetry with respect to $y=x$.

⟶ $\bar{x} = \bar{y}$

mass

$$M = \iint_D \rho(x,y)\,dA = \iint k(x^2+y^2)\,dA$$

$$\underset{\text{POLAR}}{=} \int_0^{\pi/2}\int_0^1 k \cdot r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} k\,\frac{r^4}{4}\Big|_0^1\,d\theta = \frac{\pi}{2}\cdot\frac{k}{4} = \boxed{\frac{\pi k}{8}}$$

$$M_y = \iint_D x\,\rho(x,y)\,dA = \iint x \cdot k \cdot (x^2+y^2)\,dA$$

$$\underset{\text{POLAR}}{=} \int_0^{\pi/2}\int_0^1 r\boxed{\cos\theta} \cdot k \cdot r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \boxed{\int_0^1 k r^4 \, dr} \, d\theta$$

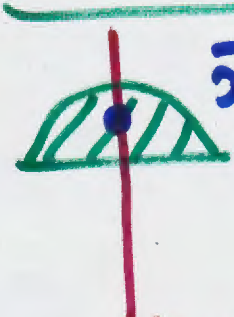
number

$$= \int_0^1 k r^4 \, dr \cdot \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= k\frac{r^5}{5}\Big|_0^1 \cdot \sin\theta \Big|_0^{\pi/2} = \boxed{\frac{k}{5}}$$

$$\bar{x} \, \cancel{\bar{y}} = \frac{k/5}{\pi k/8} = \frac{8}{5\pi} = \bar{y}$$

D:



$\bar{x} = 0$

$x = 0$

$$\rho = \sqrt{x^2 + y^2}$$

$-x$ + $+x$ give same

In general

$$\int_c^d \int_a^b \underbrace{f(x)\cdot g(y)}_{\uparrow \text{ separates}} \, dx\,dy$$

$\boxed{\text{NUMBERS}}$

ex: $\iint x \sin y \, dy\,dx \qquad \iint y^2 e^x \, dx\,dy$

<u>not</u> $\iint x+y \, dy\,dx$

$$\int_c^d g(y)\,dy \cdot \int_a^b f(x)\,dx$$