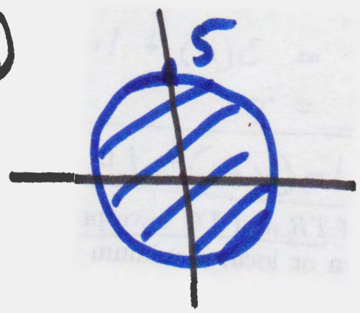
15.3 Double Integrals in Polar Coordinates

I. Polar Rectangles

R: $a \le r \le b$, $\alpha \le \theta \le \beta$ ← limits of integration

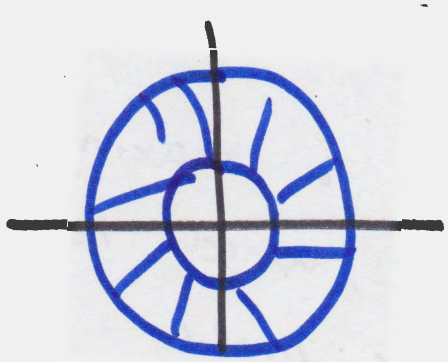
example ①



$0 \le r \le 5$

$0 \le \theta \le 2\pi$

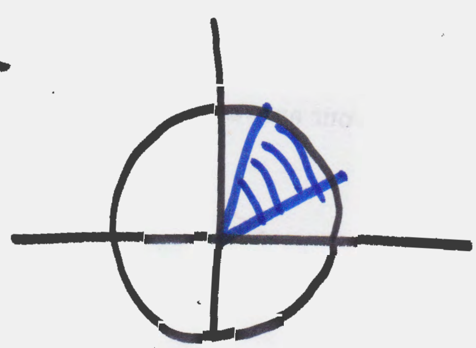
② ring



$1 \le r \le 2$

$0 \le \theta \le 2\pi$

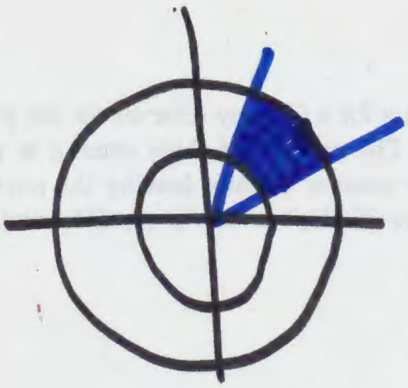
③ pizza slice



$0 \le r \le 3$

$\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

4

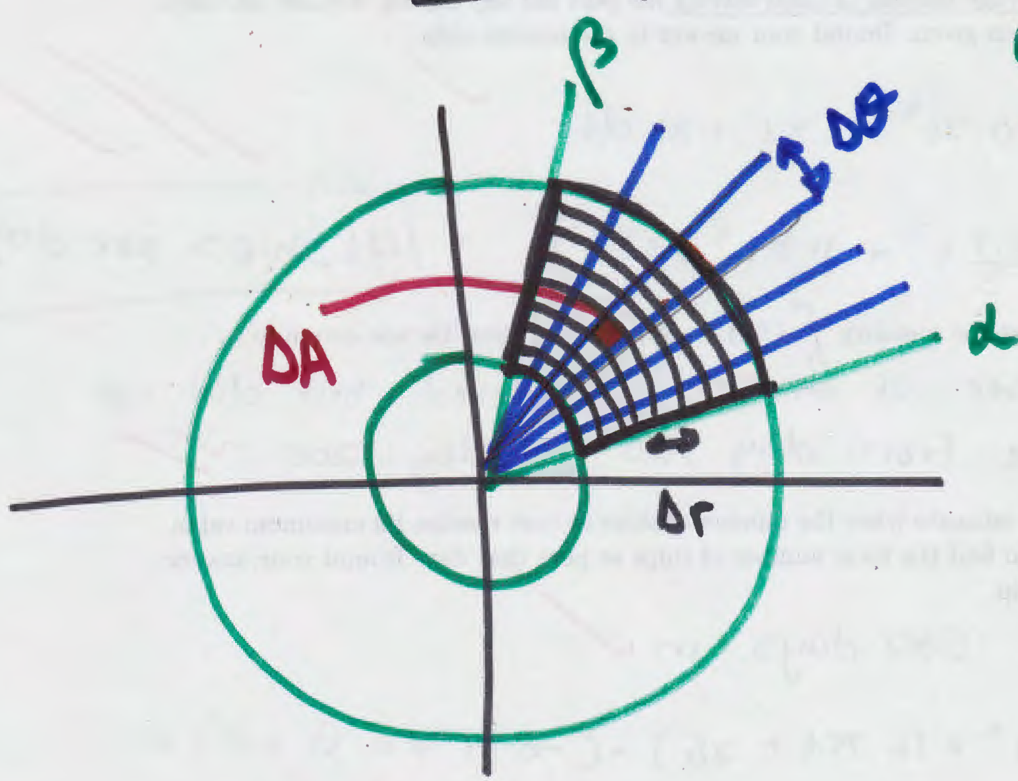


$3 \le r \le 5$

$\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

want: $\iint_{\underline{\underline{R}}} f(x,y)\, \boxed{dA}$??

$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$



$a \le r \le b$

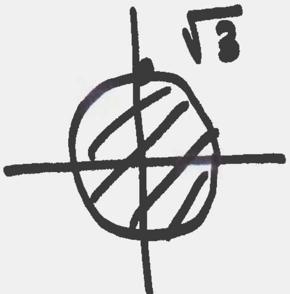
$\alpha \le \theta \le \beta$

$\frac{b-a}{m} = \Delta r$

$\frac{\beta-\alpha}{n} = \Delta\theta$

$$0 = 3 - \underbrace{x^2 - y^2} \rightarrow x^2 + y^2 = 3$$

R



$$0 \le r \le \sqrt{3}$$
$$0 \le \theta \le 2\pi$$

$$\underbrace{x^2 + y^2} = r^2$$

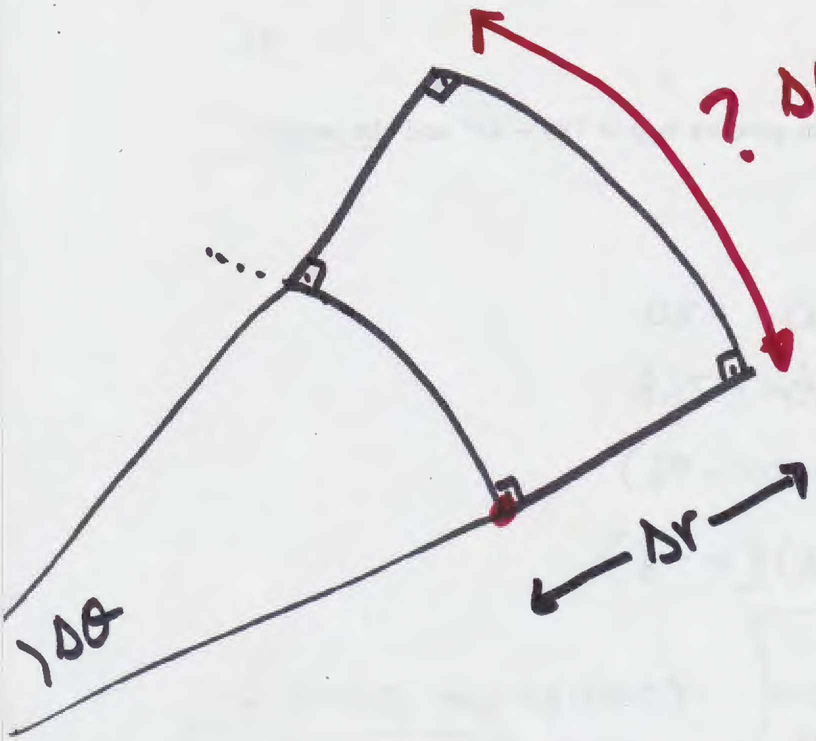
$$\int_0^{2\pi}\int_0^{\sqrt{3}} (3 - r^2)\ \underline{r}\,dr\,d\theta$$

$$u = 3 - r^2$$
$$du = -2\underline{r}\,dr$$

$$\frac{u^2}{2}$$

$$= \int_0^{2\pi} \frac{(3-r^2)^2}{2}\left(\frac{-1}{2}\right)\Bigg|_0^{\sqrt{3}} d\theta$$

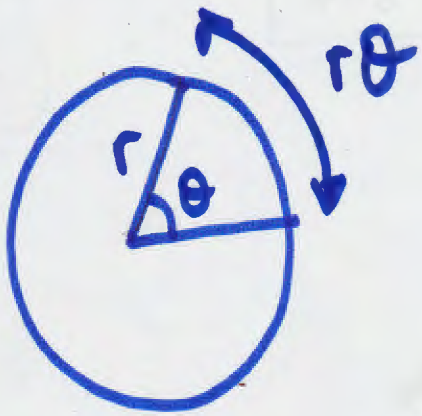
$$= \int_0^{2\pi} 0 + \frac{3^2}{4}\ d\theta = \frac{9}{4}\ \theta\Big|_0^{2\pi} = \frac{9\pi}{2}.$$



$$\Delta A \approx \Delta r \cdot r \cdot \Delta\theta$$

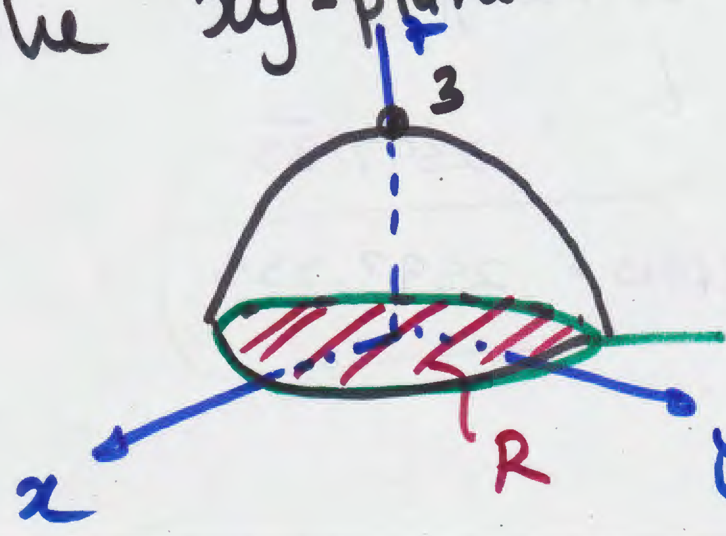
$$\boxed{dA = r\,dr\,d\theta}$$

recall:



example ① Find the volume ~~of the~~ between the paraboloid $z = 3 - x^2 - y^2$ and the xy-plane.

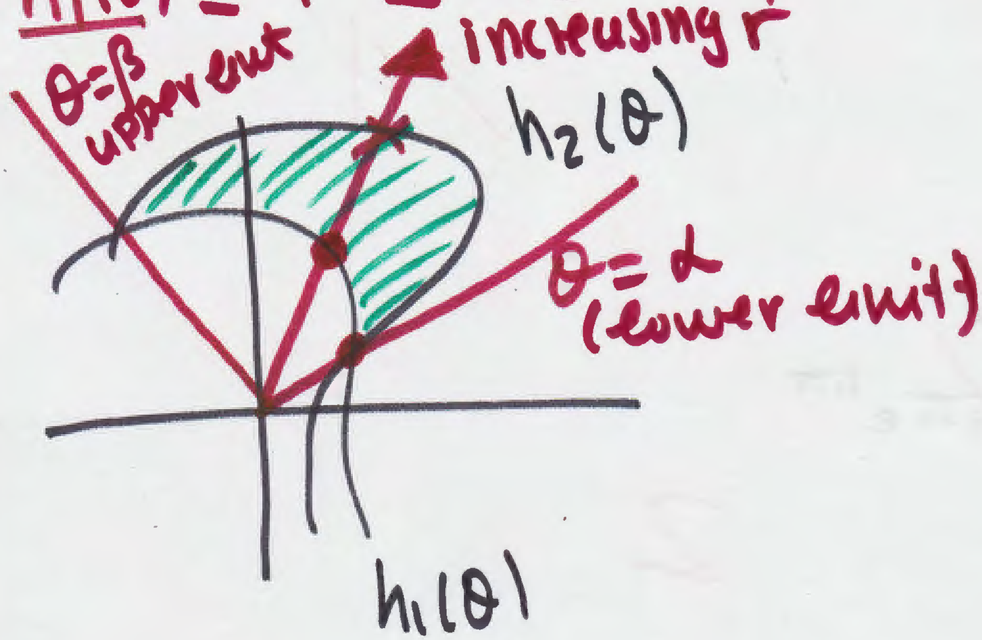
$$\iint 3 - x^2 - y^2 \, dA$$



intersection of $z = 3 - x^2 - y$ with xy-plane $z = 0$

II. More General Region

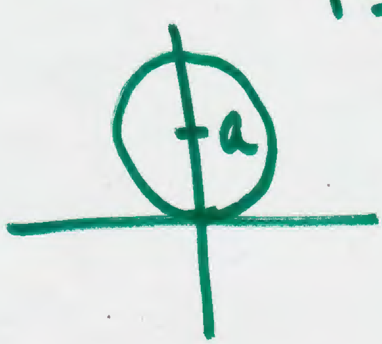
$h_1(\theta) \le r \le h_2(\theta) \qquad \alpha < \theta \le \beta$



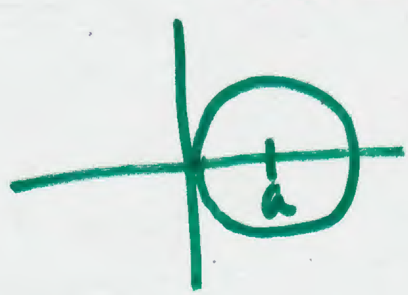
$$\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} g(r,\theta) \; r\,dr\,d\theta$$

Recall:

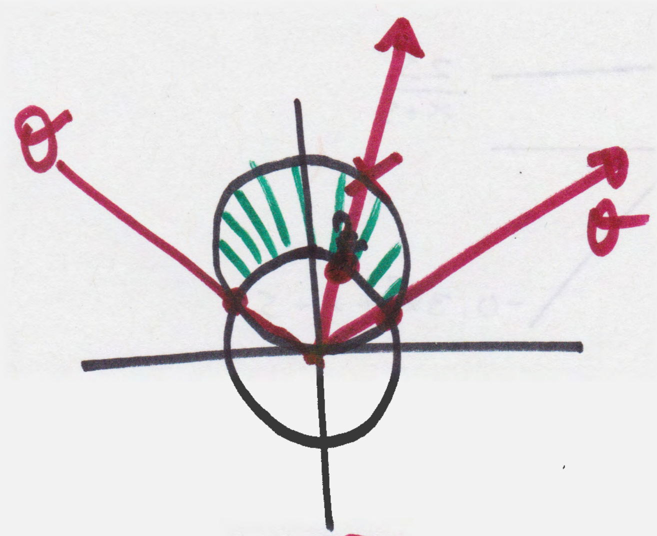
$r = 2a\sin\theta$



$r = 2a\cos\theta$



example ② Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.



intersect the circles:

$$2 = 4\sin\theta$$

$$\frac{1}{2} = \sin\theta$$

$$\theta_1 = \frac{\pi}{6} = \arcsin\frac{1}{2}$$

$$\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left.\frac{r^2}{2}\right|_{2}^{4\sin\theta} d\theta$$

$$\boxed{\iint_D dA = \text{Area of } D}$$

"sum the pieces to get the whole"

$$= \int_{\pi/6}^{5\pi/6} 8\sin^2\theta - 2 \; d\theta$$

Recall: Double angle formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_{\pi/6}^{5\pi/6} 4(1-\cos 2\theta) - 2 \, d\theta = \int_{\pi/6}^{5\pi/6} 2 - 4\cos 2\theta \, d\theta$$

$$= 2\theta - 2\sin 2\theta \Big|_{\pi/6}^{5\pi/6}$$

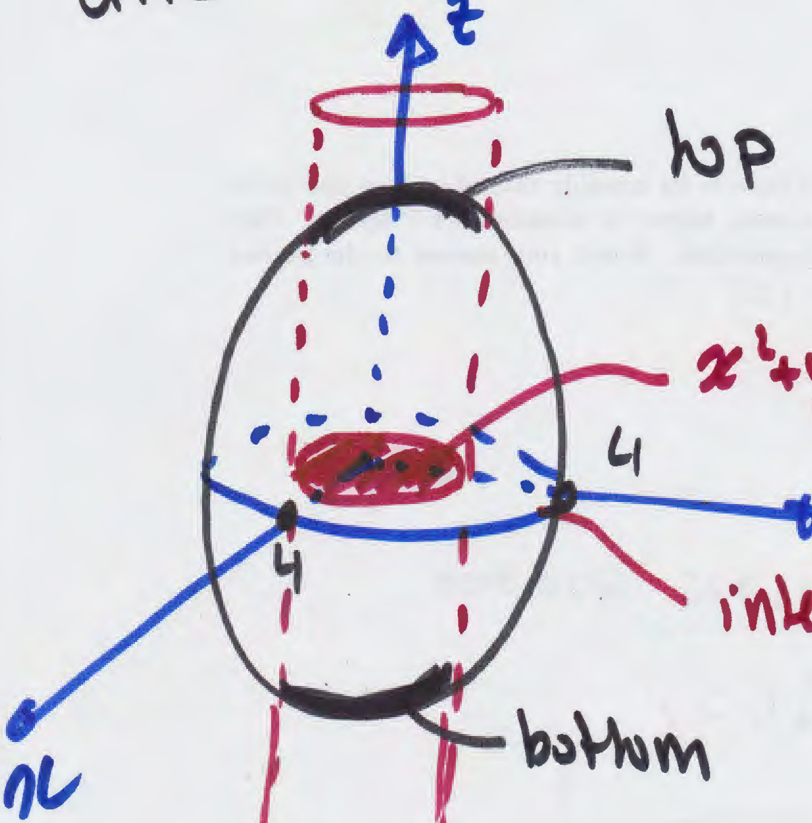
$$= 2\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - 2\left(\sin\frac{5\pi}{3} - \sin\frac{\pi}{3}\right)$$

$$= \frac{4\pi}{3} - 2\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} + 2\sqrt{3}$$

int. technique note:

$$\int \sin^3\theta \, d\theta = \int (1-\cos^2\theta)\sin\theta \, d\theta$$
$u = \cos\theta$

example ③ Find the volume inside the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$.



top $z = \sqrt{64-4x^2-4y^2}$

$x^2+y^2 \le 4 \leftarrow R : \begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array}$

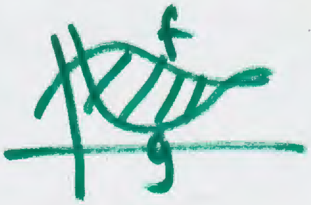
intersection of $4x^2+4y^2+z^2=64$ with xy-plane $z=0$

$\rightarrow 4x^2+4y^2=64$

$x^2+y^2=16$

Volume $= \iint$ Top surface $-$ Bottom surface dA

same idea as: M125



$\int$ top curve $-$ bottom curve $=$ area

$$\iint_R \sqrt{64-4x^2-4y^2} - \left(-\sqrt{64-4x^2-4y^2}\right) dA$$

$$2\int_0^{2\pi}\int_0^{2} \sqrt{64-4r^2} \quad \underline{\underline{r}}\, dr\, d\theta$$

$u = 64-4r^2$
$du = -8r\,dr$

$$= 2\int_0^{2\pi} \frac{2}{3}(64-4r^2)^{\frac{3}{2}}\left(\frac{-1}{8}\right)\Bigg|_0^2 \; d\theta$$

$$= 2\cdot 2\pi\cdot\left[-\frac{1}{12}\left(48^{3/2} - 512\right)\right]$$

$$= \frac{\pi}{3}\left(512 - 48^{3/2}\right)$$