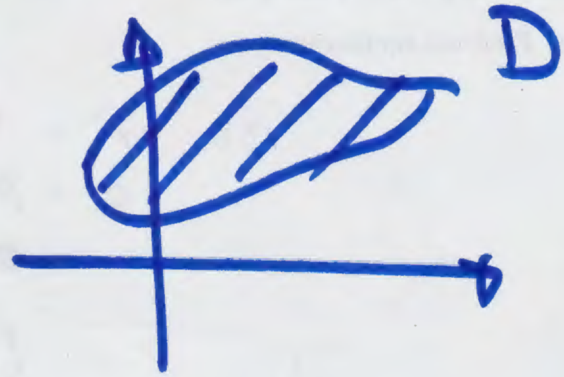
15.2 Double Integrals over General Regions

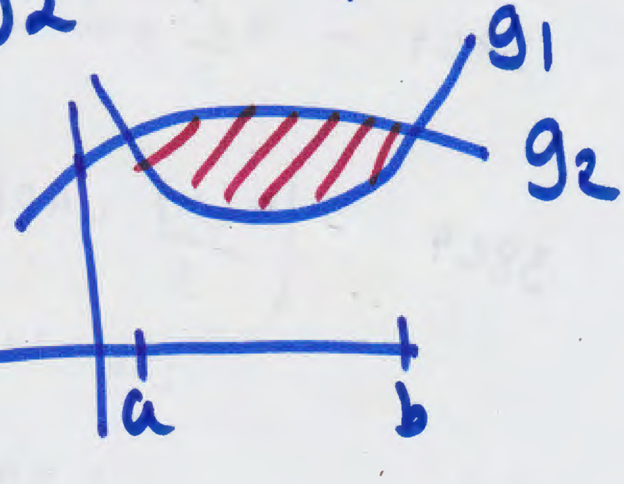
$$\iint_D f(x,y)\,dA$$

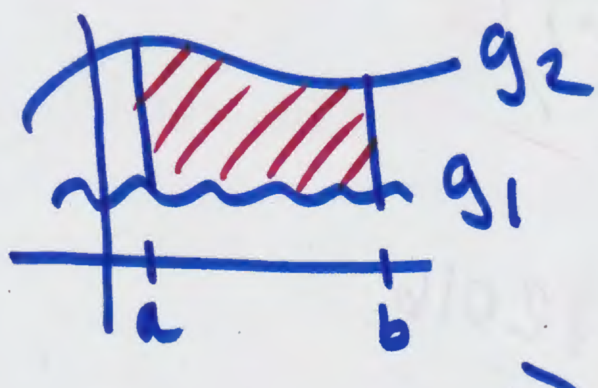


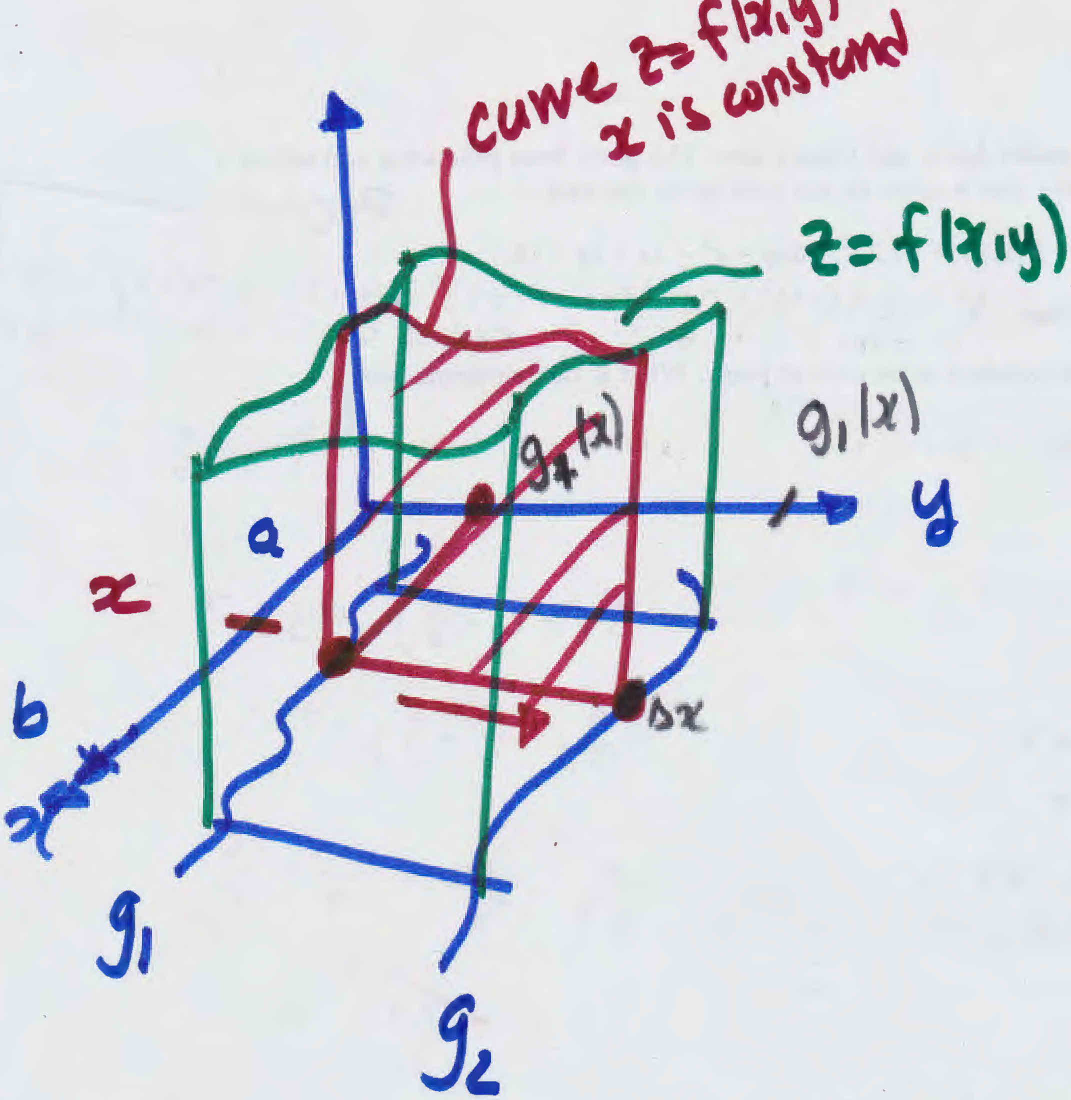
D : domain/region

There are essentially two ways to describe D

I. $g_1(x) \le y \le g_2(x)$, $a \le x \le b$







Total Volume under $z=f(x,y)$ over D

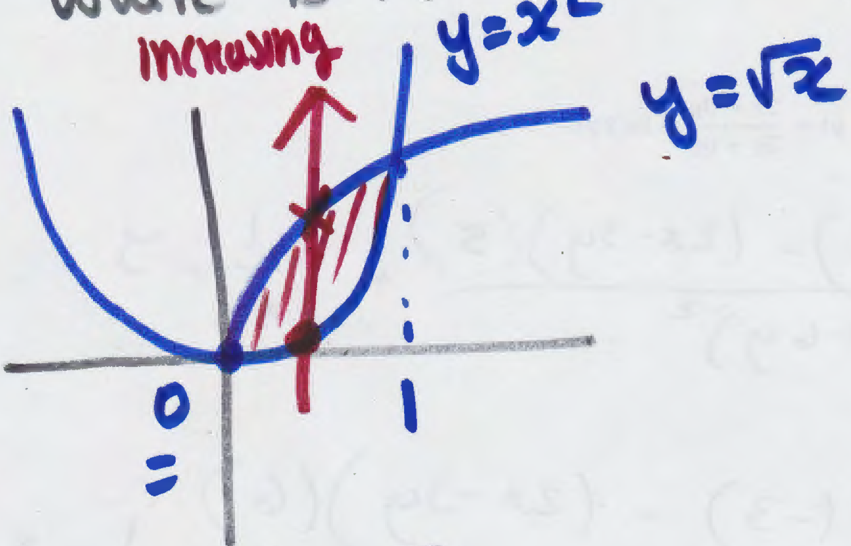
$$\int_a^b \text{"area of slice at } x\text{"}\, dx$$

area of slice at x $\displaystyle\int_{g_1(x)}^{g_2(x)} f(x,y)\, \underline{\underline{dy}}$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

example: Evaluate $\displaystyle\iint_D x^2+y \; dA$

where D is bounded by $y=\sqrt{x}$ and $y=x^2$.



$\sqrt{x}=x^2$

$x=x^4$

$x^4-x=0$

$x(x^3-1)=0$

$x=0$ or $x^3=1$

$x=1$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} x^2+y \;\; dy\,dx$$

$$= \int_0^1 x^2 y + \frac{y^2}{2} \Big|_{x^2}^{\sqrt{x}} \, dx$$

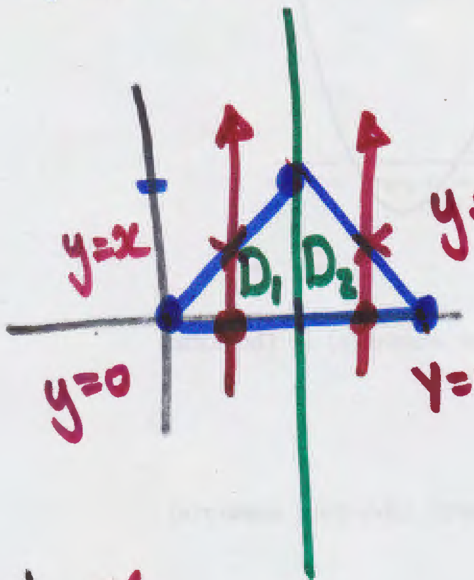
$$= \int_0^1 \left(x^{5/2} + \frac{x}{2} \right) - \left(x^4 + \frac{x^4}{2} \right) dx$$

$$\underset{\text{clean}}{=} \int_0^1 x^{5/2} + \frac{1}{2}x - \frac{3}{2}x^4 \, dx$$

$$= \frac{2}{7}x^{7/2} + \frac{1}{4}x^2 - \frac{3}{10}x^5 \Big|_0^1$$

$$= \frac{2}{7} + \frac{1}{4} - \frac{3}{10} = \frac{33}{140}.$$

ex: $\iint_D xy\, dA$ where D is the region inside the triangle with vertices $(0,0)$, $(2,0)$, and $(1,1)$.



$$\iint_{D_1} xy\, dA + \iint_{D_2} xy\, dA$$

$$= \int_0^1 \int_0^x xy \; dy\, dx + \int_1^2 \int_0^{-x+2} xy \; dy\, dx$$

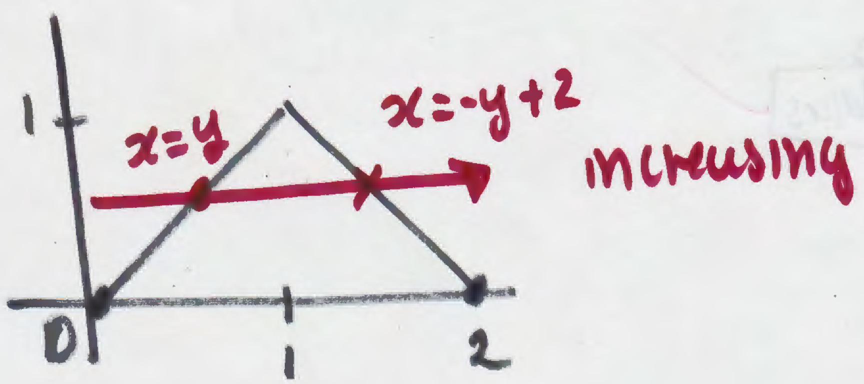
$$= \int_0^1 \frac{xy^2}{2}\Big|_0^x dx + \int_1^2 \frac{xy^2}{2}\Big|_0^{-x+2} dx$$

$$= \int_0^1 \frac{1}{2}x^3\,dx + \int_1^2 \frac{1}{2}\cdot x(-x+2)^2\,dx$$

$$= \frac{1}{8}x^4\Big|_0^1 + \frac{1}{2}\int_1^2 x^3-4x^2+4x\,dx$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2\Big|_1^2\right]$$

$= \ldots$



$$D: \quad y \le x \le -y+2, \quad 0 \le y \le 1.$$

$$\iint_D xy\, dA = \int_0^1 \int_y^{-y+2} xy \; dx\, dy$$

$$= \int_0^1 \left. \frac{x^2 y}{2} \right|_y^{-y+2} dy = \frac{1}{2}\int_0^1 (-y+2)^2 y - y^3 \, dy$$

clean

$$= \frac{1}{2}\int_0^1 \cancel{y^3} - 4y^2 + 4y - \cancel{y^3} \, dy$$

$$= \frac{1}{2}\left[-\frac{4y^3}{3} + 2y^2 \Big|_0^1 \right]$$

$$= \frac{1}{2}\left[-\frac{4}{3} + 2 \right] = \frac{1}{3}$$

In general if

$\underline{\text{II}}$. $D: h_1(y) \le x \le h_2(y),\ c \le y \le d$

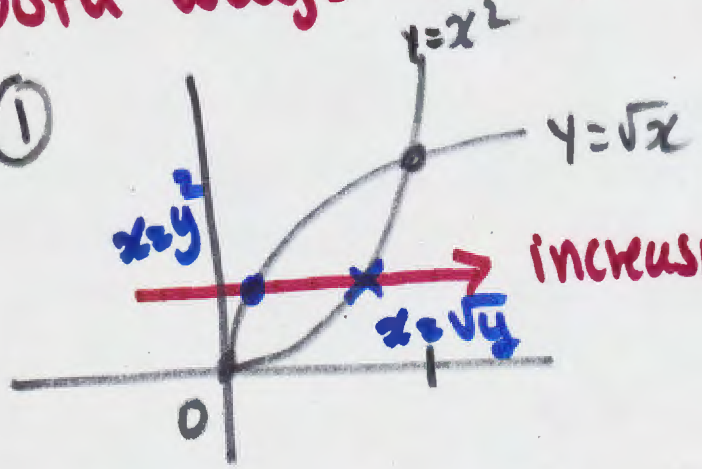
then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Notes ① Outer limits are always <u>numbers</u>.

② Some regions can be described both ways so you get a choice.

ex①



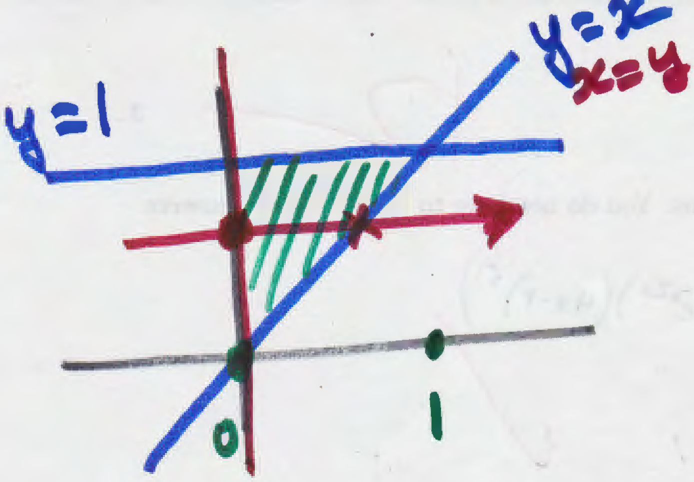
$y^2 \le x \le \sqrt{y}\ ,\ 0 \le y \le 1$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} x^2 + y \, dx \, dy$$

③ If D can be described both ways so you can do dxdy or dydx, one may be an easier integral (or one may be impossible) so switching the order of integration is an integration technique in double integrals.

ex: $$\int_0^1 \int_x^1 \sin(y^2) \, dy \, dx$$

D: $x \le y \le 1$, $0 \le x \le 1$



$$\int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$\int_0^1 x\sin(y^2) \Big|_0^y \, dy$$

$$= \int_0^1 y \sin(y^2)\, dy$$

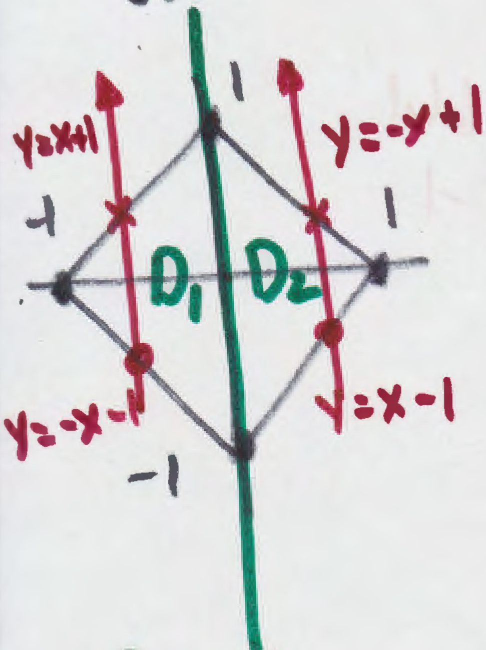
$u = y^2$

$$= \frac{1}{2}\left(-\cos(y^2)\right)\Big|_0^1 = \frac{1}{2}(-\cos 1 + \cos 0)$$

$$= \frac{1 - \cos 1}{2}.$$

④ Some regions D need to be broken into (two or more) parts in either way.

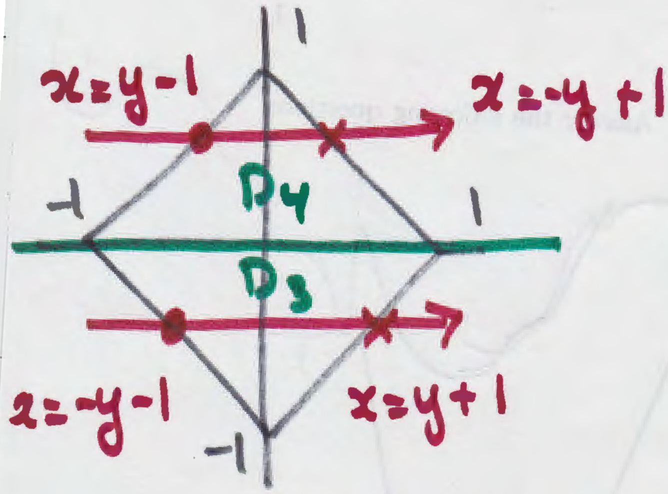
ex :



$$\iint_D f(x,y)\,dA$$

$$= \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

$$= \int_{-1}^{0}\int_{-x-1}^{x+1} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{x-1}^{-x+1} f(x,y)\,dy\,dx$$



$$\iint_D f(x,y)\,dA = \iint_{D_3} f(x,y)\,dA + \iint_{D_4} f(x,y)\,dA$$

$$= \int_{-1}^{0}\int_{-y-1}^{y+1} f(x,y)\,dx\,dy + \int_{0}^{1}\int_{y-1}^{-y+1} f(x,y)\,dx\,dy$$