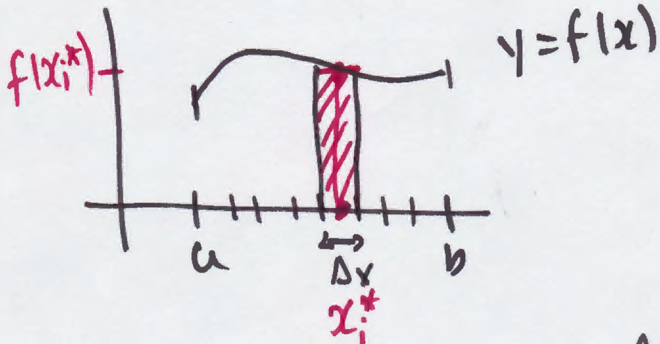
# Chapter 15, Section 15.1



Recall: The Definition of the Integral



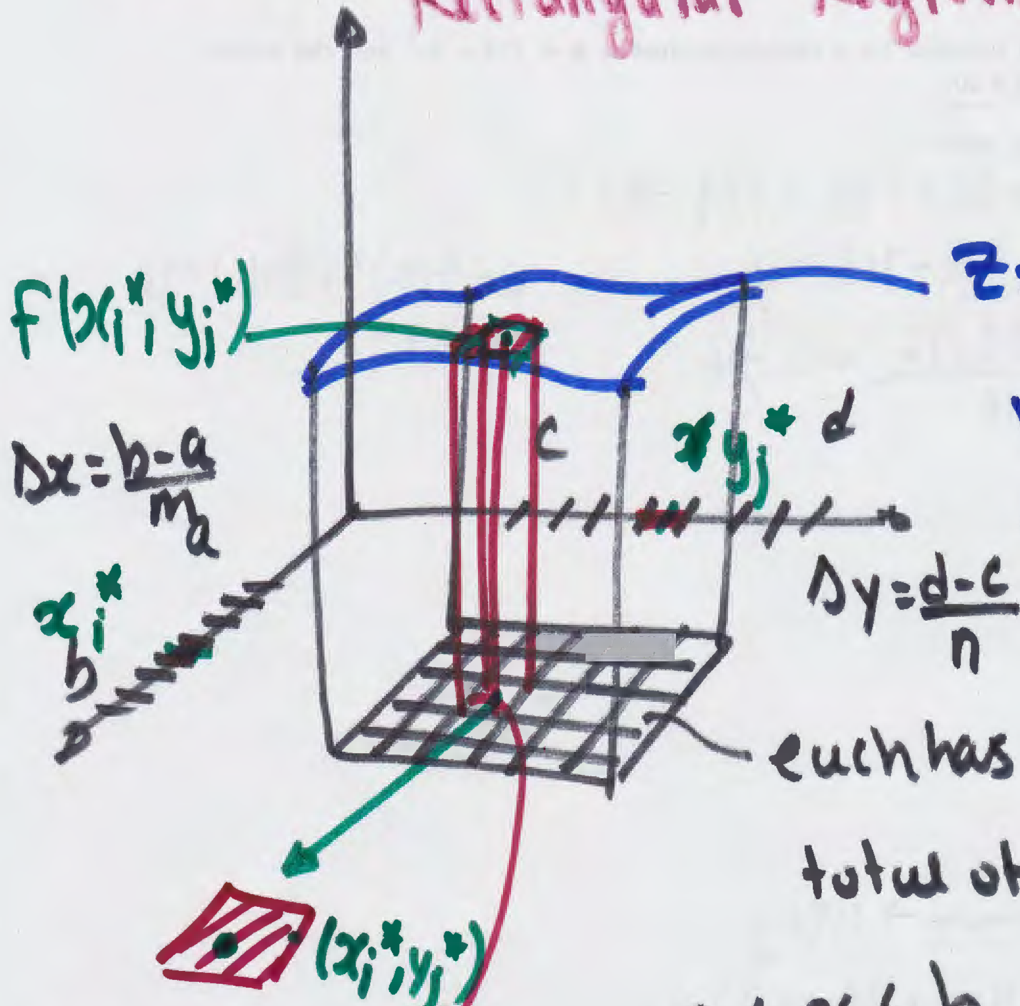

$$\Delta \text{Area} \approx \Delta x \cdot f(x_i^*)$$

$$\text{Total area} = \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Sum}}$$

$$\int_a^b f(x)\,\underline{\underline{dx}} = \lim_{\substack{n\to\infty \\ \Delta x \to 0}} \sum_{i=1}^{n} f(x_i^*)\,\underline{\underline{\Delta x}}$$

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dA$$

# 15.1 Double Integrals over Rectangular Regions




Volume under $f(x,y)$ over R?

each has area $\Delta A = \Delta x \Delta y$
$= \Delta x \Delta x$

total of <u>nm</u> rectangles

Rectangle R: $a \le x \le b$, $c \le y \le d$

$[a,b] \times [c,d]$

$$\Delta \text{Volume} \approx f(x_i^*, y_j^*) \, \Delta A$$

Then we sum the estimates

$$\text{Total Volume} \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*) \, \Delta A$$

The estimate improves with small $\Delta A$, large $n + m$.

$$\iint_R f(x,y)\,dA = \underbrace{\lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{m}}_{\text{can't do this in practice}} f(x_i^*, y_j^*)\,\Delta A$$

(volume if $f(x,y) > 0$).

If we fix $n, m$, we can estimate the integral

$$\iint_R f(x,y)\,dA \approx \sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

If $n + m$ are large

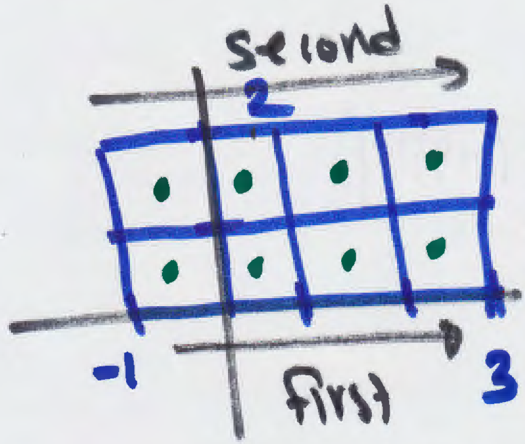
→ good estimate ☺

→ too much work ☹

example: $\displaystyle\iint_R y^2 - 2x^2 \, dA$

over the rectangle $R = [-1,3] \times [0,2]$

with $n=2$, $m=4$, using midpoints.




8 rectangles

$$\Delta A = \Delta x \Delta y = \frac{3-(-1)}{4} \cdot \frac{2-0}{2} = \underline{\underline{1}}$$

$$\iint_R y^2 - 2x^2 \, dA \approx \Big[ f\left(-\tfrac{1}{2}, \tfrac{1}{2}\right) + f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + f\left(\tfrac{3}{2}, \tfrac{1}{2}\right) + f\left(\tfrac{5}{2}, \tfrac{1}{2}\right)$$

$$+ f\left(-\tfrac{1}{2}, \tfrac{3}{2}\right) + f\left(\tfrac{1}{2}, \tfrac{3}{2}\right) + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) + f\left(\tfrac{5}{2}, \tfrac{3}{2}\right) \Big] \Delta A$$

$$= \left[ \left( \left(\tfrac{1}{2}\right)^2 - 2\left(-\tfrac{1}{2}\right)^2 \right) + \cdots + \left( \left(\tfrac{3}{2}\right)^2 - 2\left(\tfrac{5}{2}\right)^2 \right) \right] \cdot 1$$

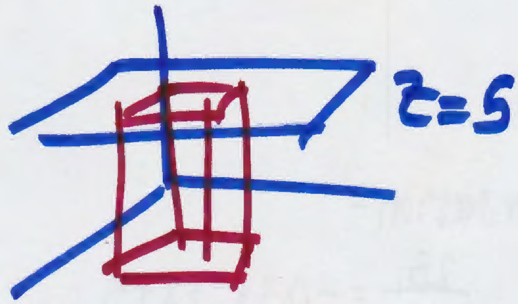
$\overset{\text{check}}{=} -26$

# Applications

① If $f(x,y) > 0$ on R then

$$\iint_R f(x,y)\,dA = \text{Volume under } z=f(x,y) \text{ over } R$$

ex: $\iint_R 5\,dA$

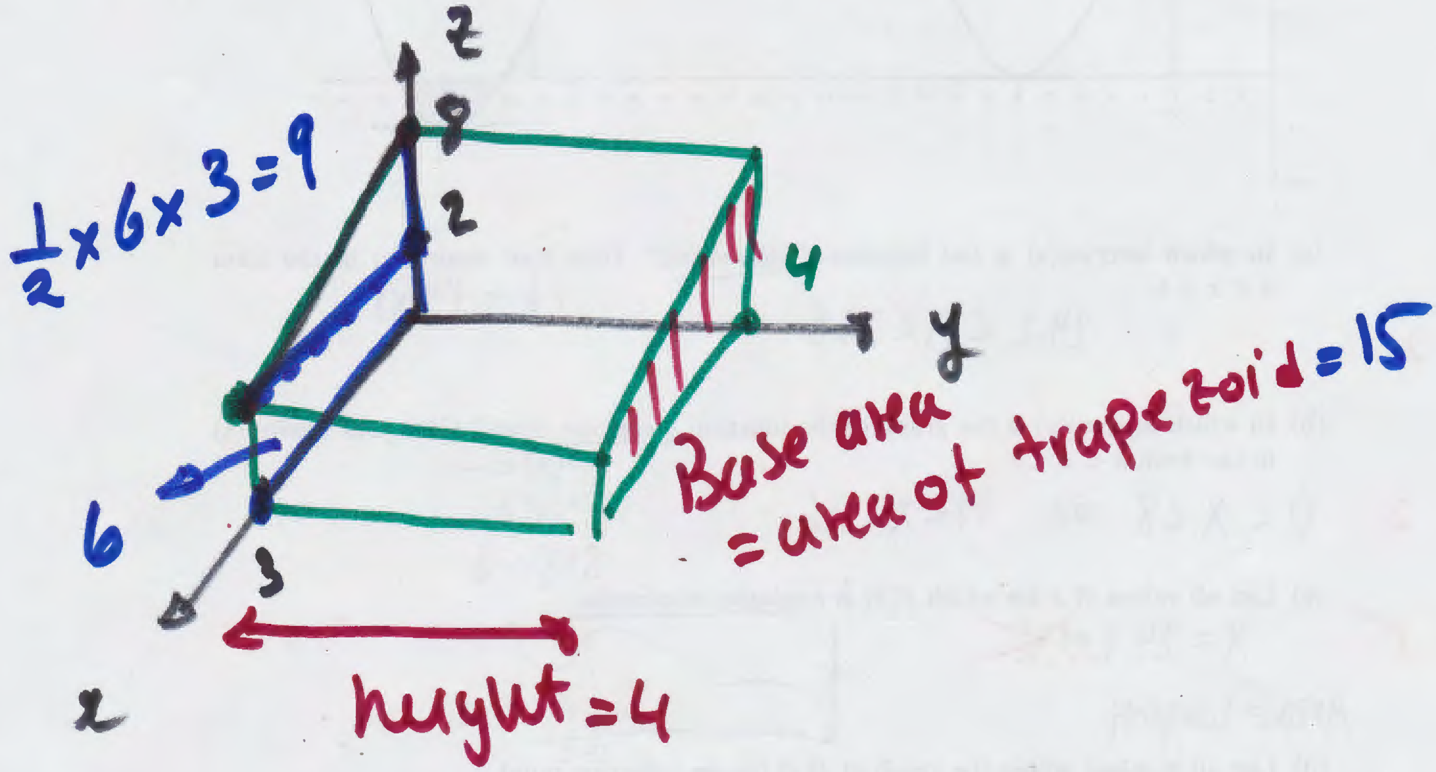



= Volume of box = 5 × area of R

In general $\iint_R c\,dA = c \cdot \text{area of } R$

ex: $\iint\limits_R 8-2x \, dA = 60$

$R = [0,3] \times [0,4]$.

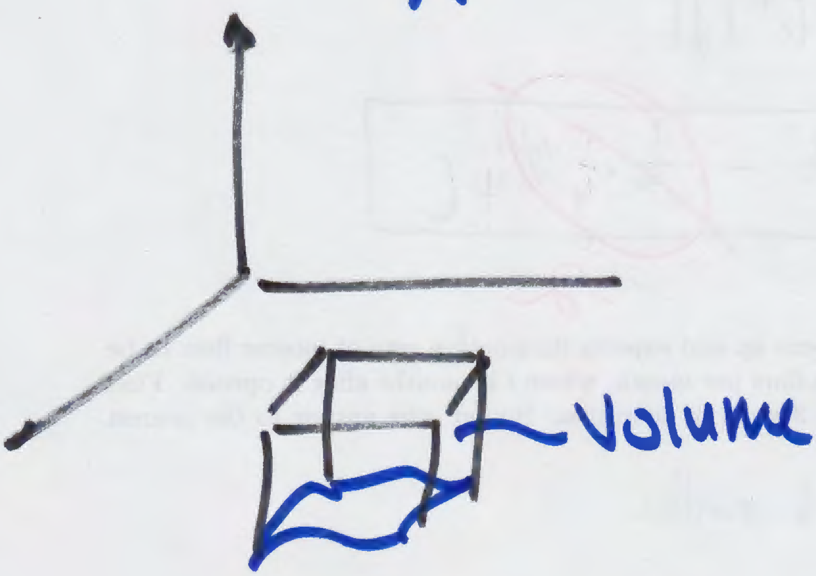



Volume = $4 \times 15 = 60$

Area of trapezoid = (average of parallel sides) · (distance between them)

② If $f(x,y) < 0$

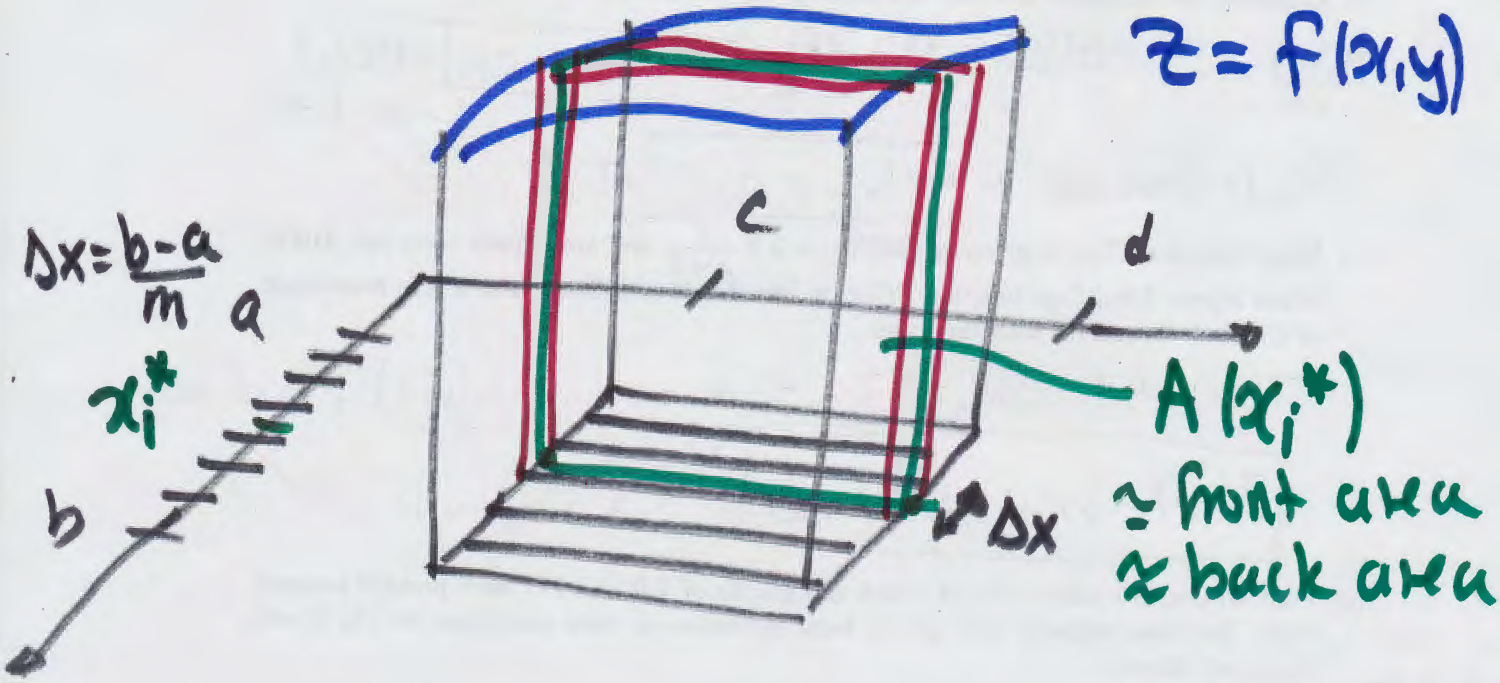
then $\iint_R f(x,y)\,dA = -$ Volume over $f$




③ Average ~~Volume~~ Value of $f(x,y)$ over $R$

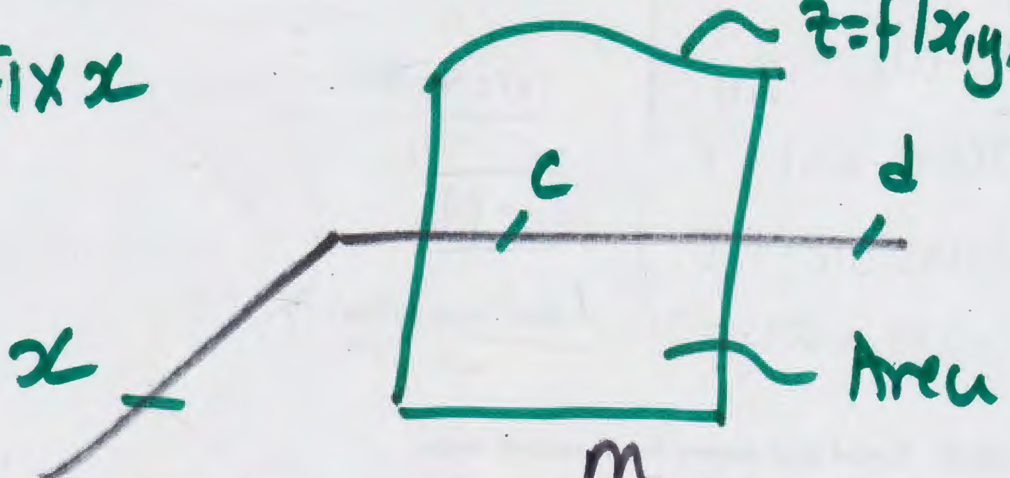
$$\frac{1}{\text{area } R} \iint_R f(x,y)\,dA$$

# Iterated Integrals




Total Volume = $\sum$ Volumes of Slices $\rightarrow A(x_i^*) \cdot \Delta x$

Fix $x$




$$\text{Total Volume} = \sum_{i=1}^{m} A(x_i^*)\,\Delta x$$

$$\int_a^b A(x)\,dx = \lim_{m\to\infty} \sum_{i=1}^{m} A(x_i^*)\,\Delta x$$

$$A(x) = \int_c^d f(x,y)\,dy$$

$x$ is fixed

$$\iint_R f(x,y)\,dA = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{x \text{ is like a constant}}\,dx$$

ex: Compute $\iint_R 2x+3y \, dA$

$R = [0,5] \times [2,4]$

$$\int_0^5 \int_2^4 2x+3y \quad dy \, dx$$

$$= \int_0^5 \left. 2xy + \frac{3y^2}{2} \right|_2^4 dx$$

$$= \int_0^5 (8x+24) - (4x+6) \quad dx$$

$$\underset{\text{clean up}}{=} \int_0^5 4x+18\,dx$$

$$= 2x^2+18x \Big|_0^5$$

$$= 50+90-0 = 140.$$

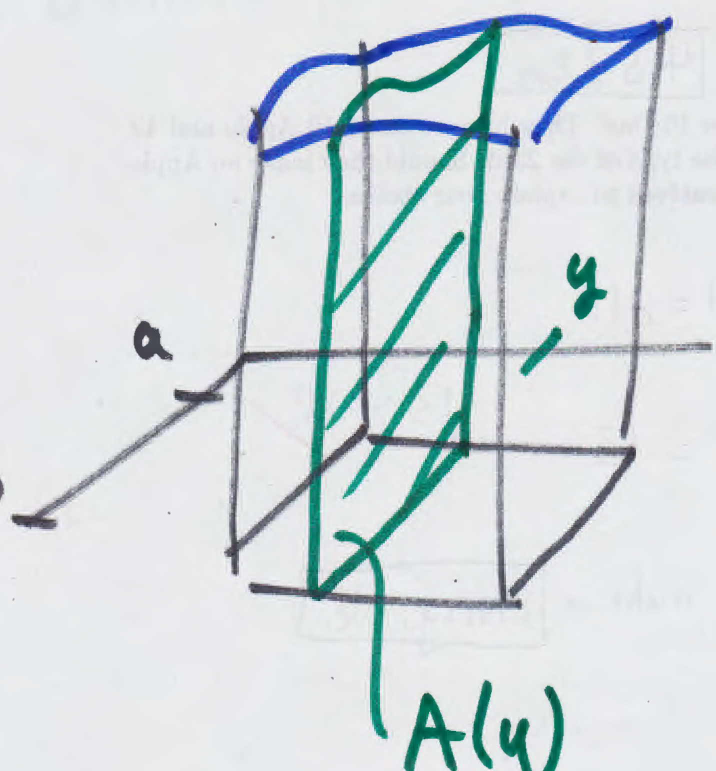
We can reverse order of integratio

$$\text{Volume} = \int_c^d A(y)\,dy$$

where

$$A(y) = \int_a^b f(x,y)\,dx$$

$\uparrow$ y fixed



$$\iint_R f(x,y) = \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{\text{treat } y \text{ like a constant}}\,dy$$

ex: $$\iint_R \cos(x+5y)\,dA \quad , R=[0,\pi]\times[0,\tfrac{\pi}{2}]$$

$$\int_0^{\pi/2}\int_0^{\pi} \cos(x+5y)\,dx\,dy$$

$$=\int_0^{\pi/2} \sin(x+5y)\Big|_0^{\pi}\,dy$$

$$= \int_0^{\pi/2} \sin(\underbrace{\pi+5y}_{u=\pi+5y}) - \sin(\underbrace{5y}_{u=5y})\, dy$$

$$= \left. \frac{-\cos(\pi+5y)}{5} + \frac{\cos(5y)}{5} \right|_0^{\pi/2}$$

$$= \left( \cancelto{0}{\frac{-\cos(\pi+\frac{5\pi}{2})}{5}} + \cancelto{0}{\frac{\cos\frac{5\pi}{2}}{5}} \right) - \left( \frac{-\cos\pi}{5} + \frac{\cos 0}{5} \right)$$

$$= \frac{-1}{5} - \frac{1}{5} = -\frac{2}{5}.$$