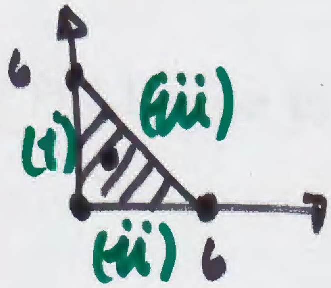
example: Find the absolute max. and absolute min. of

$$f(x,y) = 4x + 6y - x^2 - y^2$$

on the (triangular) domain $x \geq 0,\ y \geq 0,\ x + y \leq 6.$

Domain



I. Look for critical points of f inside D.

$$f_x = 4 - 2x = 0 \longrightarrow x = 2$$

$$f_y = 6 - 2y = 0 \longrightarrow y = 3$$

$$\boxed{(2,3)}$$

(check CP is in your domain. If not, toss it.)

II. On the boundary (triangle)

(i) $x=0$ (y-axis) $\boxed{0 \le y \le 6}$

$$f(0,y) = \boxed{6y - y^2 = g(y)}$$

$$g'(y) = 6 - 2y = 0 \rightarrow \boxed{y=3}$$

endpoints $\boxed{y=0}$ $\boxed{y=6}$

$$\boxed{(0,3),\ (0,0),\ (0,6)}$$

(ii) $y=0$ (x-axis) $\underline{\underline{0}} \le x \le \underline{\underline{6}}$

$$f(x,0) = 4x - x^2 = g(x)$$

$$g'(x) = 4 - 2x = 0 \rightarrow x = \underline{\underline{2}}$$

$(0,0)$ $\boxed{(2,0),\ (6,0)}$

already listed

(iii) $x+y=6 \longrightarrow y=6-x, \underbrace{0\le x\le 6}$

$$f(x,6-x)=\underbrace{4x+6(6-x)-x^2-(6-x)^2}=g(x)$$

(you can simplify g before differentiating)

$$g'(x)=4+6(-1)-2x-2(6-x)(-1)=0$$

$$4-6-2x+12-2x=0$$

$$10=4x$$

$$\frac{5}{2}=\frac{10}{4}=x$$

$\boxed{\left(\frac{5}{2},\frac{7}{2}\right)}$ $(0,6)$ $(6,0)$

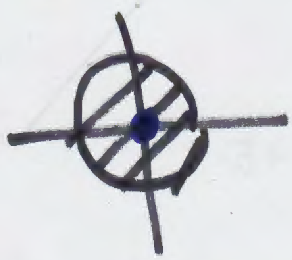
already listed already listed

	CP for f	CNs for boundary pieces			endpoints of boundary pieces		
points	$(2,3)$	$(0,3)$	$(2,0)$	$\left(\frac{5}{2},\frac{7}{2}\right)$	$(0,6)$	$(6,0)$	$(0,0)$
$f(x,y)$	13	9	4	$\frac{25}{2}$	0	−12	0
	MAX					min	

ex② Find the min. + max. of

$$f(x,y) = 4y^3 + x^4 + 5$$

on $x^2 + y^2 \leq 1$.

Domain  Boundary: $x^2+y^2=1$

I. Critical Point(s) of f

$f_x = 4x^3 = 0 \rightarrow x = 0$

$f_y = 12y^2 = 0 \rightarrow y = 0$

$\boxed{(0,0)}$

II Working on the boundary

we want to reduce the problem to the one variable case. We can do that by eliminating x or y.

OPTION 1 Eliminate y

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

$-1 \le x \le 1$

(i) $f(x, \sqrt{1-x^2}) = 4(1-x^2)^{3/2} + x^4 + 5$

(ii) $f(x, -\sqrt{1-x^2}) = -4(1-x^2)^{3/2} + x^4 + 5$

2 functions to deal with

$\sqrt{\quad}$

OPTION 2 eliminate x

$$x^2 = 1 - y^2$$

Both (i) $x = -\sqrt{1-y^2}$ + (ii) $x = \sqrt{1-y^2}$ will give the same function because of x^2

$$f(\pm\sqrt{1-y^2}, y) = 4y^3 + (1-y^2)^2 + 5 = g(y) \quad -1 \le \cancel{x}\, y \le 1.$$

$$g'(y) = 12y^2 + 2(1-y^2)(-2y) = 0$$

$$4y\left[3y - (1-y^2)\right] = 0$$

$$4y(3y - 1 + y^2) = 0$$

$$y = 0 \quad \text{OR} \quad y = \frac{-3 \pm \sqrt{9 - 4(1)(-1)}}{2}$$

$$= \frac{-3 \pm \sqrt{13}}{2}$$

$$\cancel{y = \frac{-3-\sqrt{13}}{2}} < -1 \qquad y = \frac{-3+\sqrt{13}}{2} \approx 0.303$$

$$\boxed{(\pm 1, 0) \quad \left(\pm\sqrt{1-\left(\frac{-3+\sqrt{13}}{2}\right)^2}, \frac{-3+\sqrt{13}}{2}\right)}$$

points

points	$(0,0)$	$(\pm 1, 0)$	$\left(\pm\sqrt{1-\left(\frac{-3+\sqrt{13}}{2}\right)^2}, \frac{-3+\sqrt{13}}{2}\right)$
	CP for f	endpoints for boundary	critical numbers for boundary
$f(x,y)$	5	6	≈ 5.936
	↑ min	↑ MAX	

—o—

OPTION 3 go parametric on the unit circle.

$x = \cos t \qquad y = \sin t \quad , \; 0 \le t \le 2\pi$

$f(\cos t, \sin t) = 4\sin^3 t + \cos^4 t + 5 = g(t)$

g' would be ugly.

example: Find the min + max values of $f(x,y) = 2x + 3y + 7$ on $x^2 + y^2 \leq 1$.

I. Critical Points

$f_x = 2$
$f_y = 3$ $\rightarrow$ no CPs

II Boundary $x^2 + y^2 = 1$.

$x = \cos t \quad y = \sin t \qquad 0 \leq t \leq 2\pi$

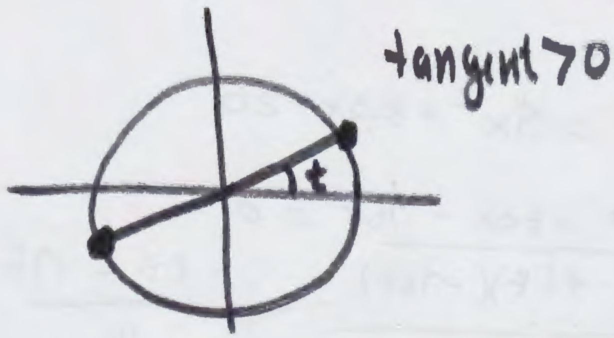
$f(\cos t, \sin t) = 2\cos t + 3\sin t + 7 = g(t)$

$g'(t) = -2\sin t + 3\cos t = 0$

$3\cos t = 2\sin t$

$\frac{3}{2} = \tan t$

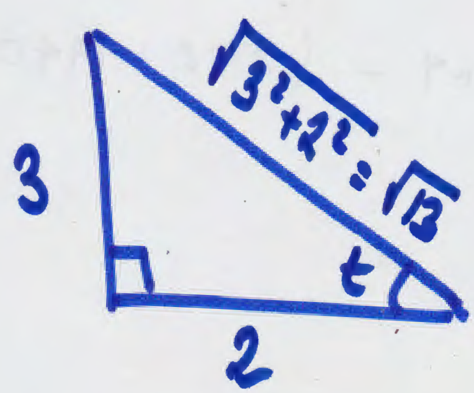
$t = \tan^{-1}\left(\frac{3}{2}\right)$, $t = \pi + \tan^{-1}\left(\frac{3}{2}\right)$



critical #s boundary

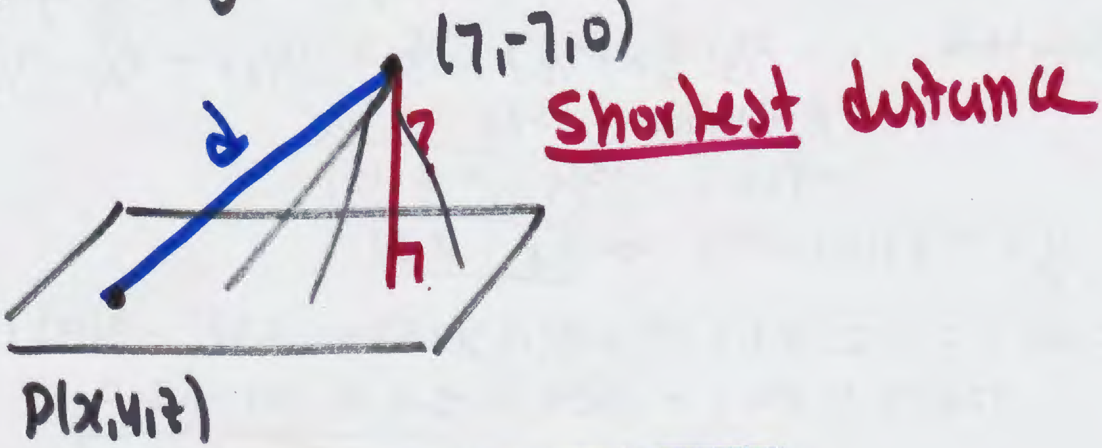
	"endpoint" $t=0$, $t=2\pi$	$t=\tan^{-1}\left(\frac{3}{2}\right)$	$t=\pi+\tan^{-1}\left(\frac{3}{2}\right)$
points	$(1,0)$	$\left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$	$\left(\frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right)$
$f(x,y)$	9	$\sqrt{13}+7 > 10$ MAX	$-\sqrt{13}+7 \approx 3.4$ min

$\cos\left(\tan^{-1}\left(\frac{3}{2}\right)\right) = \frac{2}{\sqrt{13}}$



Applied Optimization Problems

ex: Use calculus to find the distance from the point $Q(7,-7,0)$ to the plane $2x+3y+z=49$.



$$d = \sqrt{(x-7)^2+(y+7)^2+z^2}$$

i.e. want to minimize $\sqrt{(x-7)^2+(y+7)^2+z^2}$

$$z = 49-2x-3y$$

so minimize $\sqrt{(x-7)^2+(y+7)^2+(49-2x-3y)^2}$

to make computations easier I will minimize d^2:

$f(x,y) = (x-7)^2 + (y+7)^2 + (149-2x-3y)^2$

Domain: all of xy-plane

Back to Monday's lecture (no boundary)

once you find CP, make sure it is a <u>min</u> by checking f_{xx} + D.

$0 = f_x$

$0 = f_y$