# Optimization in one variable

$y = f(x)$

① $f'(x) = 0$ (or $f'$ not defined)

→ critical numbers (candidates for max/min)

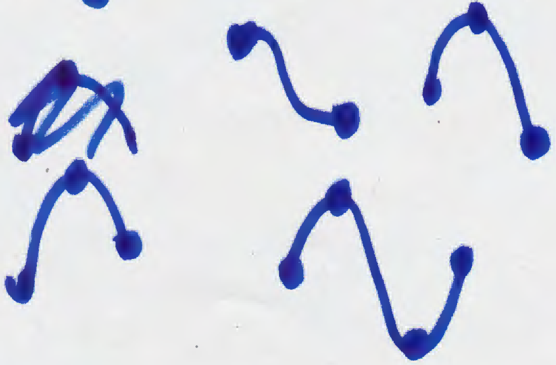
Domain $a \le x \le b$

endpoints

compare $f(a), f(b)$, $f(\text{critical } \#)$

pick max + min.

Max + min are guaranteed.



other domain (eg. all #s or $x > 0$)

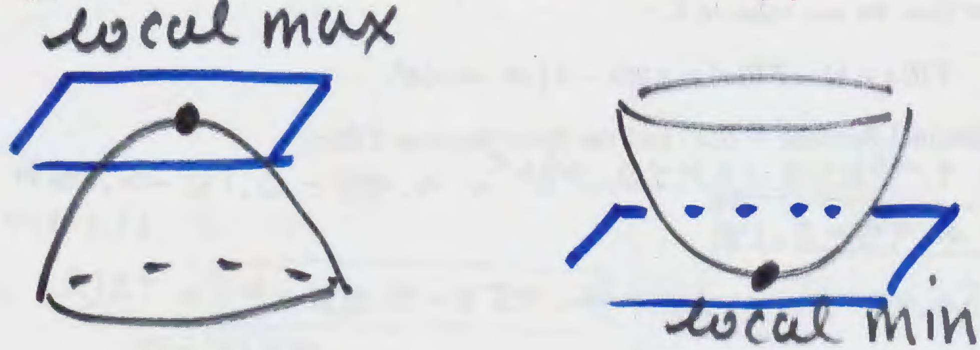
we look at $f''$

If $f'' > 0$, local min

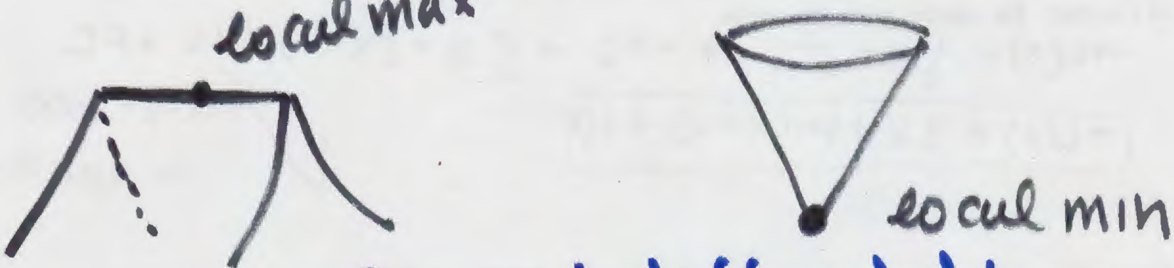
If $f'' < 0$ local max

neither

# 14.7 Maximum + minimum values for $f(x,y)$




tangent plane is parallel to the $xy$-plane $\Rightarrow f_x = f_y = 0$




$f$ is not differentiable

Theorem: If $f(x,y)$ has a local max or a local min at $(x,y)$ then

$$f_x = f_y = 0$$

(or $f$ is not differentiable at $(x,y)$)

Such points ($f_x = f_y = 0$ OR $f$ not diff.) are called CRITICAL POINTS.

example : Find the critical points of $f(x,y) = x^2 + y^4 + 2xy$.

$$f_x = 2x + 2y = 0 \longrightarrow y = -x$$

$$f_y = 4y^3 + 2x = 0$$

$$4(-x)^3 + 2x = 0$$

$$-4x^3 + 2x = 0$$

$$-2x(2x^2 - 1) = 0$$

$x = 0$ OR $2x^2 = 1$

$\Downarrow$ $\qquad$ $x = \pm\sqrt{\frac{1}{2}}$

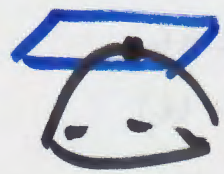
$y = 0$ $\qquad$ $\Downarrow$

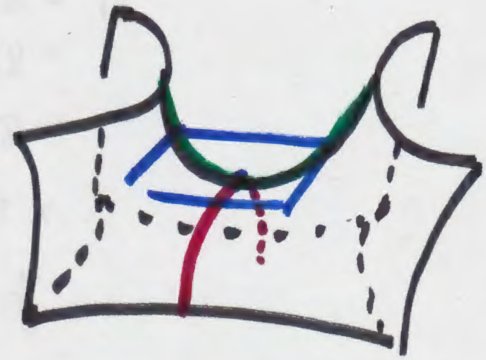
$y = \mp\sqrt{\frac{1}{2}}$

$$(0,0),\ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

How do I decide if a CP gives a ~~min~~

min OR max OR saddle





or some other picture?

## SECOND DERIVATIVES TEST

We define the DISCRIMINANT

$$D = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

(a) If $D<0$, then it's a SADDLE.

(b) If $D>0$, then:

If $f_{xx}>0$, then MIN

If $f_{xx}<0$, then MAX

If $f_{xx}=0$, then ?

(c) If $D=0$, then ???

In previous example:

$f_{xx}=2$ $\quad f_{xy}=2$ $\quad f_{yy}=12y^2$

$D(0,0)=2\cdot 0-2^2=-4$ SADDLE

$D\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)=2\cdot 12\left(\frac{-1}{\sqrt{2}}\right)^2-4=8$ $\quad f_{xx}=2>0$ MIN

$D\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=2(12)\left(\frac{1}{\sqrt{2}}\right)^2-4=8$ $\quad f_{xx}=2>0$ MIN example: Find the local max, local min, saddle points of

$$f(x,y) = x^4 - 2x^2 + y^3 - 3y$$

## CRITICAL POINTS

$$f_x = 4x^3 - 4x = 0$$

$$4x(x^2-1) = 0$$

$$x = 0 \quad \text{OR} \quad x = \pm 1$$

$$f_y = 3y^2 - 3 = 0$$

$$3(y-1)(y+1) = 0$$

$$y = \pm 1$$

I have 6 critical points

(0,1) (0,-1) (1,1) (1,-1) (-1,1) (-1,-1)

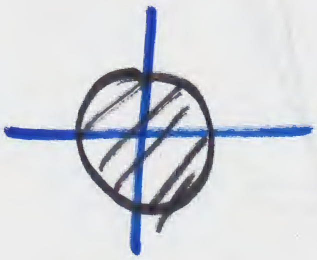
# SECOND DERIVATIVES

$f_{xx} = 12x^2 - 4$ $\quad f_{xy} = 0$ $\quad f_{yy} = 6y$

| CPs | $f_{xx}$ | $f_{xy}$ | $f_{yy}$ | D | verdict |
|---|---|---|---|---|---|
| (0,1) | -4 | 0 | 6 | -24 | saddle |
| (0,-1) | -4 | 0 | -6 | 24 | max |
| (1,1) | 8 | 0 | 6 | 48 | min |
| (1,-1) | 8 | 0 | -6 | -48 | saddle |
| (-1,1) | 8 | 0 | 6 | 48 | min |
| (-1,-1) | 8 | 0 | -6 | -48 | saddle |

# Absolute Maximum + Absolute Minimum Values

A **closed** set (in the $xy$-plane) is one which contains its boundary.



$x^2+y^2 \leq 1$

boundary: $x^2+y^2=1$

CLOSED

$x^2+y^2<1$

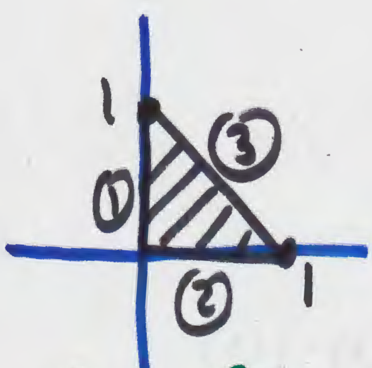
boundary: $x^2+y^2=1$

not CLOSED

$x^2+y^2 \geq 1$

boundary: $x^2+y^2=$

closed




$x \geq 0, y \geq 0, x+y \leq 1$

Boundary is made up of 3 pieces:

① $x=0$, $0 \leq y \leq 1$

② $y=0$, $0 \leq x \leq 1$

③ $x+y \leq 1$, $0 \leq x \leq 1$

CLOSED

$x+y \leq 0.2$

CLOSED

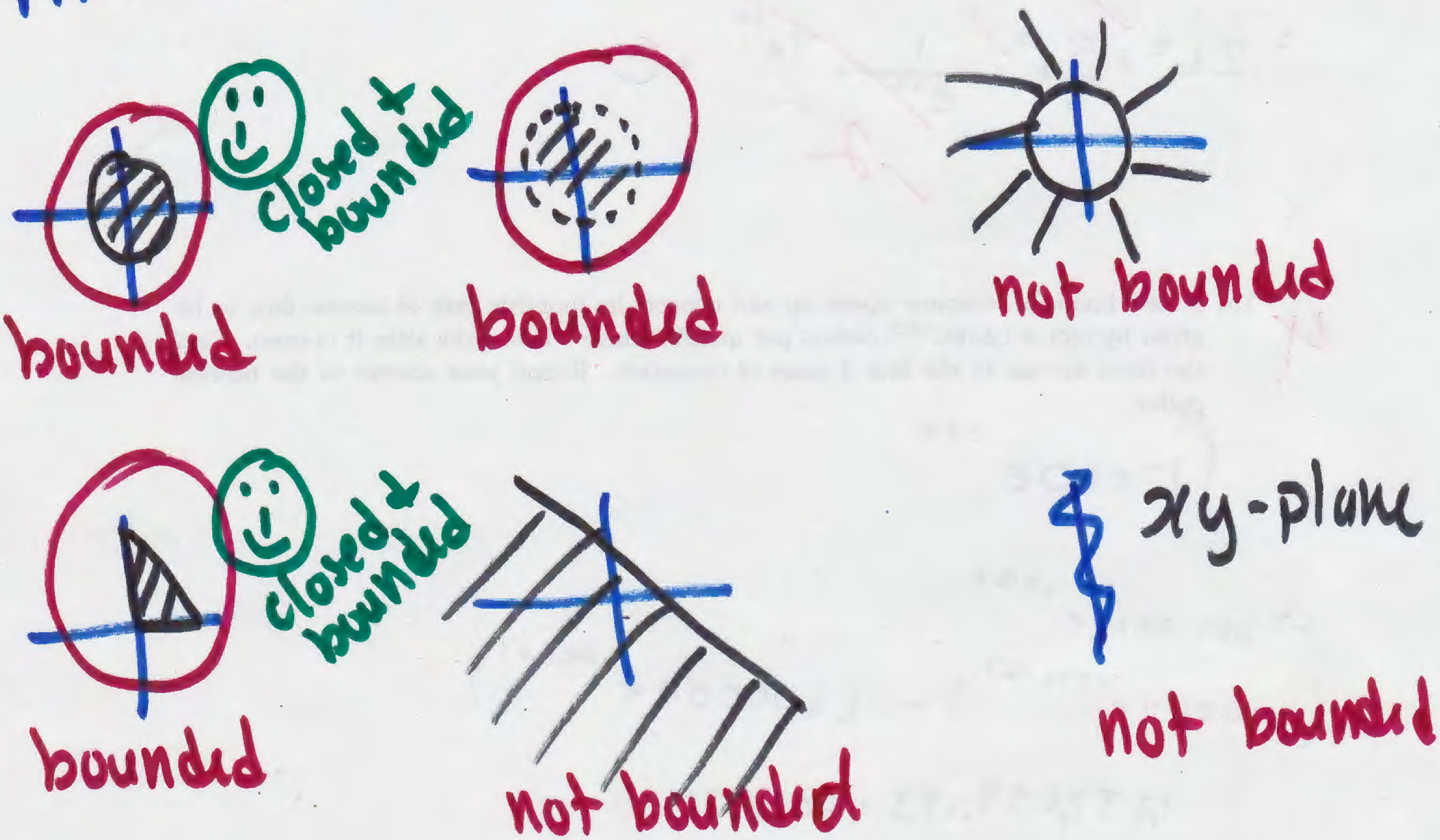
boundary $x+y=0.2$

all of $xy$-plane

CLOSED

If the boundary is defined with $\leq, \geq$ the set will be closed.

A bounded set is one which fits in a disk that is large enough.



Theorem: If f is differentiable, then it is continuous.

Theorem: If $f(x,y)$ is continuous on a closed and bounded set on xy-plane, then f has an absolute minimum and an absolute maximum either

- at a critical point

OR

- on its boundary.