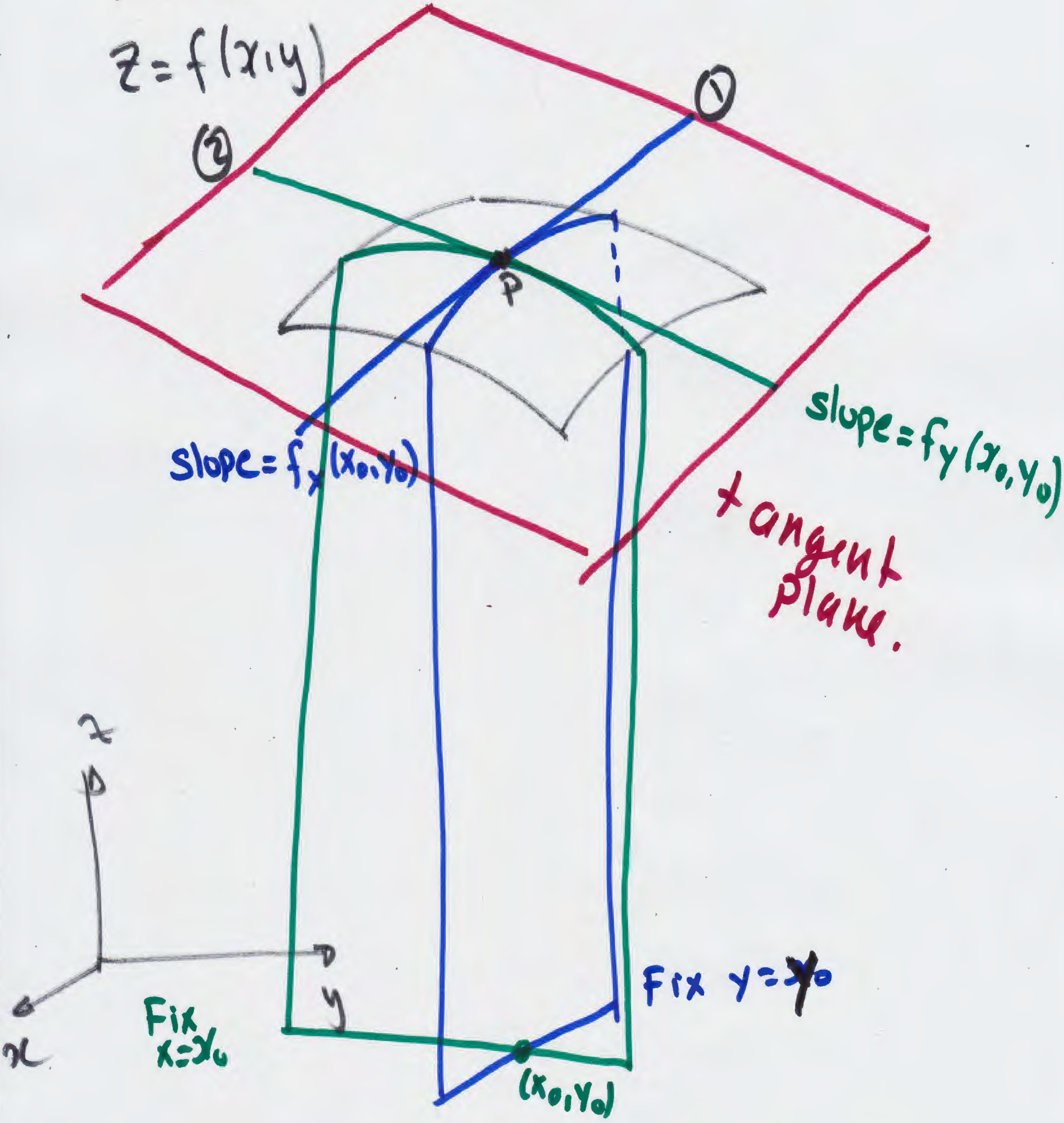
14.4 Tangent Planes + Linear Approximation



Tangent plane has the point $P(x_0, y_0, f(x_0,y_0))$

z_0

equation:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

If $C=0$, then we have a "generalized cylinder" giving a plane parallel to the z-axis. In this case, we say f is not differentiable at $x=x_0$, $y=y_0$.

~~If~~ So $C \neq 0$, divide equation by C

$$\frac{A}{C}(x-x_0)+\frac{B}{C}(y-y_0)+z-z_0=0$$

$$z = f(x_0,y_0) \underbrace{\boxed{-\frac{A}{C}}}_{\text{rename } a}(x-x_0) \underbrace{\boxed{-\frac{B}{C}}}_{\text{rename } b}(y-y_0)$$

$$z = f(x_0,y_0) + a(x-x_0) + b(y-y_0)$$

Intersect this tangent plane with the plane $y = y_0$.

Algebraically I get

$$z = f(x_0, y_0) + a(x - x_0) \quad , \quad y = y_0$$

Geometrically I get the (blue) tangent line ①

SLOPE = $f_x(x_0, y_0)$

Now, intersect this tangent plane with the plane $x = x_0$.

Algebraically : $z = f(x_0, y_0) + b(y - y_0) \, , \, x = x_0$

Geometrically: we get the (green) tangent line ②

SLOPE = $f_y(x_0, y_0)$

The equation of the tangent plane to $z = f(x,y)$ at $(x,y) = (x_0, y_0)$ is

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

example: Find the equation of the tangent plane to $z = xy + \ln(xy)$ at the point where $x = 2$ and $y = \frac{1}{2}$.

$$f(x,y) = xy + \ln(xy)$$

$$f_x(x,y) = y + \frac{1}{xy} \cdot y = y + \frac{1}{x}$$

$$f_y(x,y) = x + \frac{1}{xy} \cdot x = x + \frac{1}{y}$$

$$f\left(2,\tfrac{1}{2}\right) = 2 \cdot \frac{1}{2} + \ln\left(2 \cdot \tfrac{1}{2}\right) = 1 + \ln 1 = 1$$

$$f_x\left(2,\tfrac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

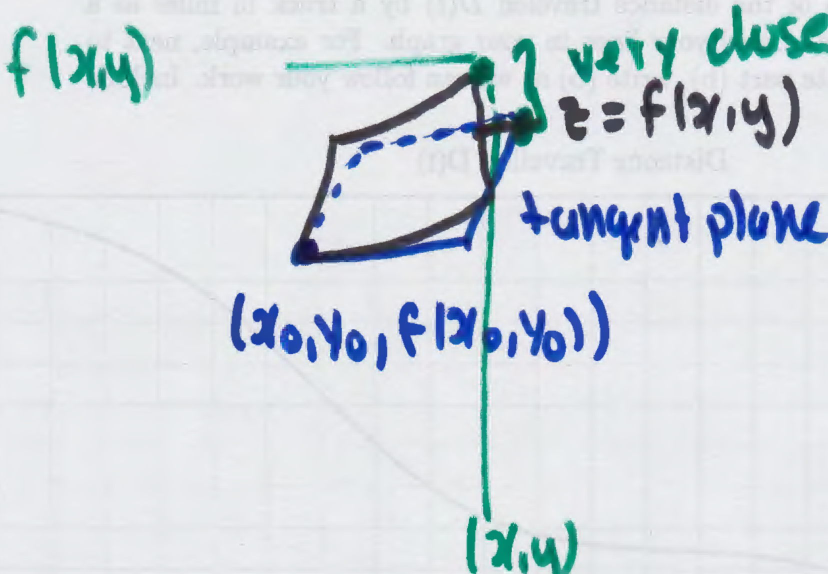
$$f_y\left(2,\tfrac{1}{2}\right) = 2 + \frac{1}{\frac{1}{2}} = 4$$

Tangent plane

$$z = 1 + 1(x - 2) + 4\left(y - \tfrac{1}{2}\right)$$

clean up: $z = x + 4y - 3$.

Linear Approximation



IDEA: for $z=f(x,y)$ + tangent plane $z=ax+by+c$ are <u>very</u> close near the point of tangency.

$$\boxed{f(x,y) \approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}$$

as long as (x,y) is <u>close</u> to (x_0,y_0)

In my example: $f(x,y) = xy + \ln(xy)$

tangent plane $z = x + 4y - 3$
at $x = 2, y = \frac{1}{2}$

If I want $f(1.9, 0.49)$

$$f(x,y) \approx x + 4y - 3$$

$$f(1.9, 0.49) \approx 1.9 + 4(0.49) - 3$$

$$= 1.9 + 1.96 - 3$$

$$= 0.86$$

example: The point $(1,1,1)$ is on the surface

$$xy^3 + 3yz^4 + 5x^2z = 9.$$

(a) Compute z_x + z_y by implicit differentiation.

Differentiate wrt x

$$y^3 + 3y\underbrace{4z^3 z_x}_{\text{chain}} + \underbrace{10xz + 5x^2 z_x}_{\text{product}} = 0$$

$$(12yz^3 + 5x^2)\, z_x = -y^2 - 10xz$$

$$z_x = \frac{-y^2 - 10xz}{12yz^3 + 5x^2}$$

Differentiate wrt y

$$3xy^2 + \underbrace{3z^4 + 3y\cdot \overbrace{4z^3 z_y}^{\text{chain}}}_{\text{product}} + 5x^2 z_y = 0$$

$$(12yz^3 + 5x^2)\, z_y = -3xy^2 - 3z^4$$

$$z_y = \frac{-3xy^2 - 3z^4}{12yz^3 + 5x^2}$$

(b) Write equation of tangent line at $(1,1,1)$. $z_0 = 1$

~~zxuuuu~~ $z_x = \dfrac{-1-10}{12+5} = -\dfrac{11}{17}$

$z_y = \dfrac{-3-3}{12+5} = -\dfrac{6}{17}$

Tangent plane

$$z = 1 - \frac{11}{17}(x-1) - \frac{6}{17}(y-1)$$

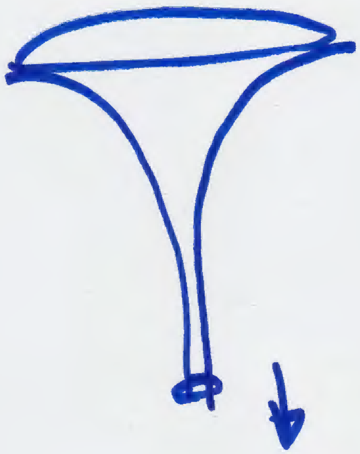
(c) Use linear approximation to approximate c if $(0.9, 1.1, c)$ is on this surface.

$$c \approx 1 - \frac{11}{17}(0.9-1) - \frac{6}{17}(1.1-1)$$

What does it mean to say $f(x,y)$ is differentiable?

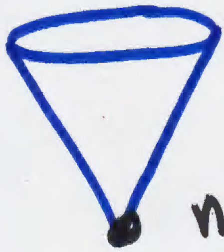
① It must be continuous at (x_0, y_0) so no holes or skips.

ex: $f(x,y) = \ln(x^2+y^2)$ is not differentiable at $(0,0)$

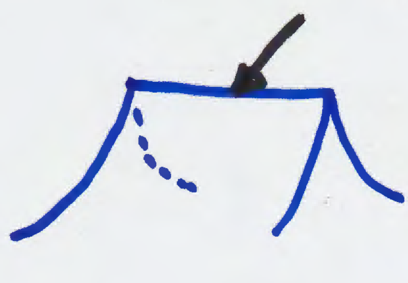


② no "corners", "vertices" or "pinches"

$f(x,y) = \sqrt{x^2+y^2}$



not diff. at $(0,0)$



not diff. anywhere on this "pinch"

Differentiable at (x_0, y_0) means when I zoom in I see a plane (tangent plane)

& the (tangent) plane should not be "vertical" or parallel to z-axis.

$z = \sqrt{1-x^2-y^2}$ is not differentiable on $z=0$

$x^2+y^2+z^2=1$

$2x+2zz_x=0$

$z_x = -\frac{x}{z}$