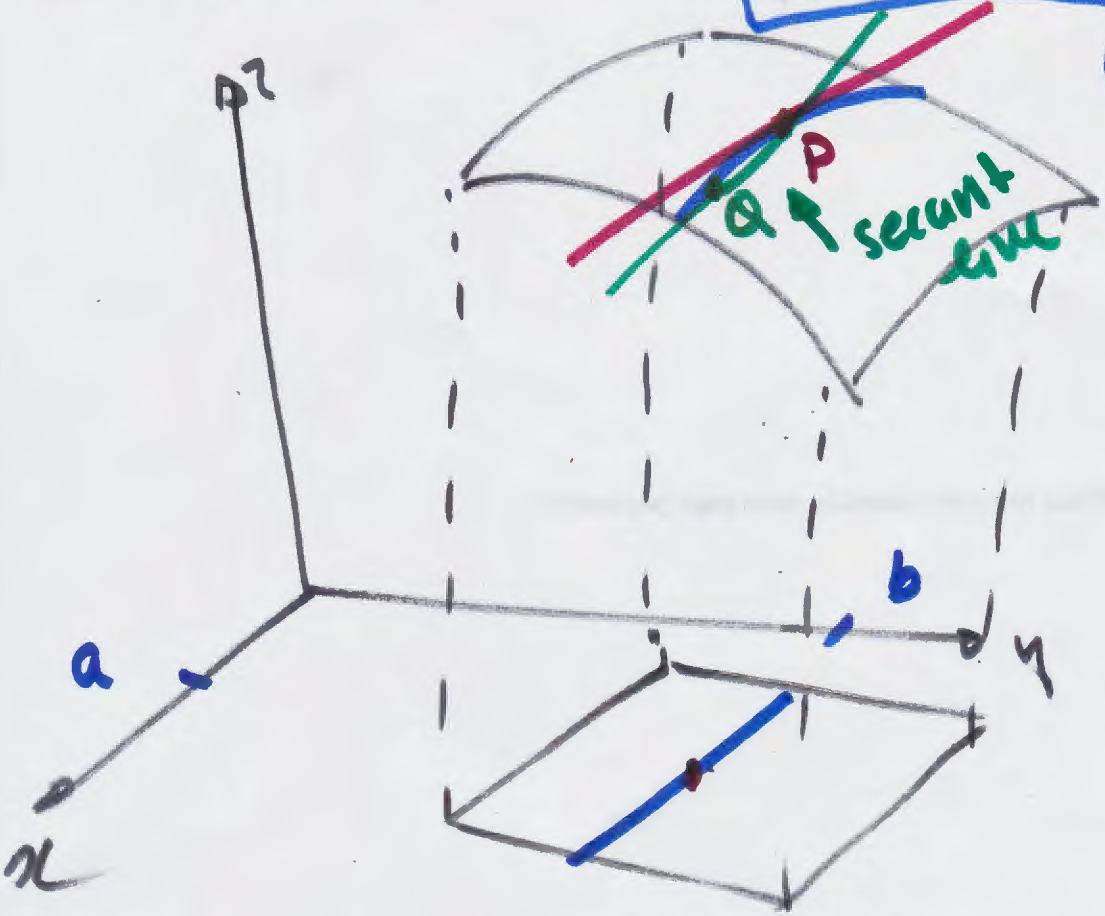
14.3 Partial Derivatives

$z = f(x,y)$ - graph is surface

$P\ (a, b, f(a,b))$

$\boxed{f_x(a,b) = \text{slope}}$ as $\frac{\Delta z}{\Delta x}$

curve $z = f(x,b)$ on the plane $y = b$



$f_x(a,b)$ - <u>Fix</u> $y = b$

Pick Q very close to P : $Q(a+h, b, f(a+h, b$

slope of secant line as $\frac{\Delta z}{\Delta x}$:

$$\frac{f(a+h,b) - f(a,b)}{a+h-a}$$

To bring Q ~~clos~~ close to P

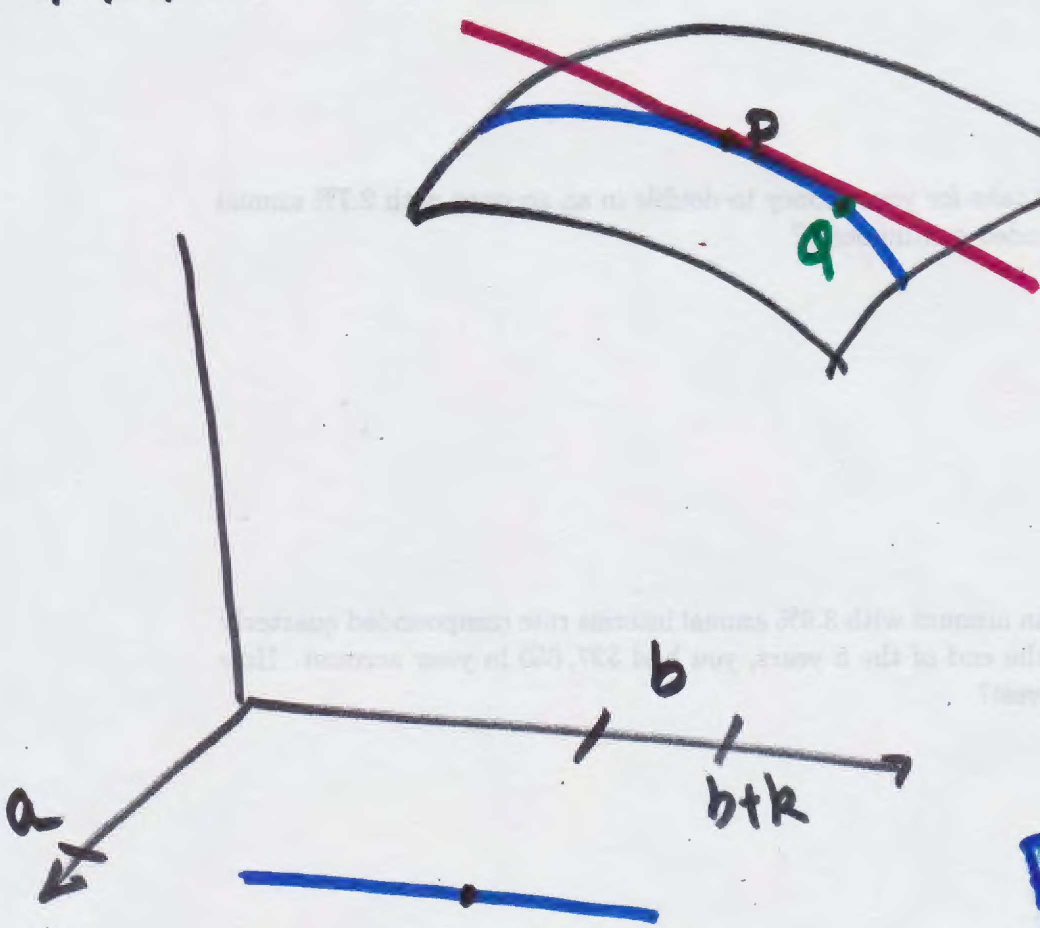
so the slope of secant → slope of tangent

we take $h \to 0$

$$\lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$

P(a,b,f(a,b))

Q(a,b+k,f(a,b+k))



slope: $f_y(a,b)$

computed as $\frac{\Delta z}{\Delta y}$

$x=a$

For $f_y(a,b)$

we fix $x=a$

$$f_y(a,b) = \underbrace{\lim_{k\to 0} \overbrace{\frac{f(a,b+k)-f(a,b)}{k}}^{\text{slope of secant } PQ}}_{\text{slope of tangent at } P}$$

Computing f_x + f_y

ALL rules of differentiation are valid

to compute f_x we treat y like a constant

to compute f_y we treat x like a constant examples: ① $f(x,y) = 2+3x+4y$

$$f_x = 0 + 3 + 0$$

② $f(x,y) = 4x^2 + \underbrace{5xy}_{(5y)x} + 6y^3$

$$f_x = 8x + 5y + 0$$

③ $f(x,y) = \dfrac{x^2+2e^y}{3x+xy}$

der. of the top with respect to x

der. of bottom with respect to

$$f_x = \frac{(2x+0)(3x+xy) - (x^2+2e^y)(3+y)}{(3x+xy)^2}$$

quotient rule

④ $f(x,y) = \sin(e^{xy} + x^3)$

$$f_x = \cos(e^{xy}+x^3) \cdot (e^{xy} \cdot y + 3x^2)$$

chain rule

derivative of inside with respect to x

same examples with f_y
(almost)

① $f(x,y) = 2 + 3x + 4y$

$f_y = 0 + 0 + 4$

② $f(x,y) = 4x^2 + (5x)y + 6y^3$

$f_y = 0 + 5x + 18y^2$

$f_y(1,3) = 5 + 18(9) = 167$

③ $f(x,y) = (x^3 + 4e^y)(\sqrt{y} + \ln(1+x))$

$y^{1/2}$

$f_y = (0 + 4e^y)(\sqrt{y} + \ln(1+x)) + (x^3 + 4e^y)\left(\frac{1}{2\sqrt{y}} + 0\right)$

product

④ $f(x,y) = \tan(e^{xy} + x^3)$

$f_y = \sec^2(e^{xy} + x^3)(e^{xy} \cdot x + 0)$

chain

ex: $f(x,y,z) = \underline{x}^2 y + y^3 z + 4\underline{x}\cos z$

(treat z & y like constants)

$f_x = 2xy + 0 + 4\cos z = \frac{\partial f}{\partial x}$

$f_y = x^2 + 3y^2 z + 0 = \frac{\partial f}{\partial y}$

$f_z = 0 + y^3 + 4x(-\sin z) = \frac{\partial f}{\partial z}$

—o—

Notation

f' $\underline{f_x}$, $f_x(x,y)$, $f_x(1,3)$, $\underline{z_x}$, $z_x(x,y)$

$\underline{f_y}$, $f_y(x,y)$, $f_y(1,3)$, $\underline{z_y}$, $z_y(x,y)$

—o—

$\frac{df}{dx}$ $\frac{\partial f}{\partial x}$ " del f by del x "

$\cancel{\partial f}$ $\frac{\partial z}{\partial x}$, $\frac{\partial}{\partial x} f(x,y)$, $\frac{\partial}{\partial x}(x^2+2xy)\Big|_{(1,2)} = 2+4=6$

ex: $\frac{\partial}{\partial x}(x^2+2xy) = 2x+2y$

ex: $f(x,y) = 2x^2y^3 + 4y^5 + 3x^4 + xy$

$f_x = 4xy^3 + 12x^3 + y$

$f_y = 6x^2y^2 + 20y^4 + x$

$(f_x)_x = f_{xx} = 4y^3 + 36x^2$

$(f_x)_y = f_{xy} = 12xy^2 + 1$

$(f_y)_x = f_{yx} = 12xy^2 + 1$

$(f_y)_y = f_{yy} = 12x^2y + 80y^3$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem: $f_{xy} = f_{yx}$

if f is "nice".

All <u>our</u> functions are "nice".

ex: Implicit differentiation

$\cancel{5z}$ $$5x^2+4y^2+3z^2=1$$

Find z_x + z_y without solving for z first.

when computing z_x, y is like a constant and z is really a function ($z=f(x,y)$)

with respect to x:

$$10x + 0 + 3\cdot 2z\cdot z_x = 0$$

chain rule

Solve for z_x:

$$z_x = \frac{-10x}{6z}$$

p. derivative with respect to y:

$$0 + 8y + 6z \cdot z_y = 0$$

because of chain rule

so $z_y = \frac{-8y}{6z}$