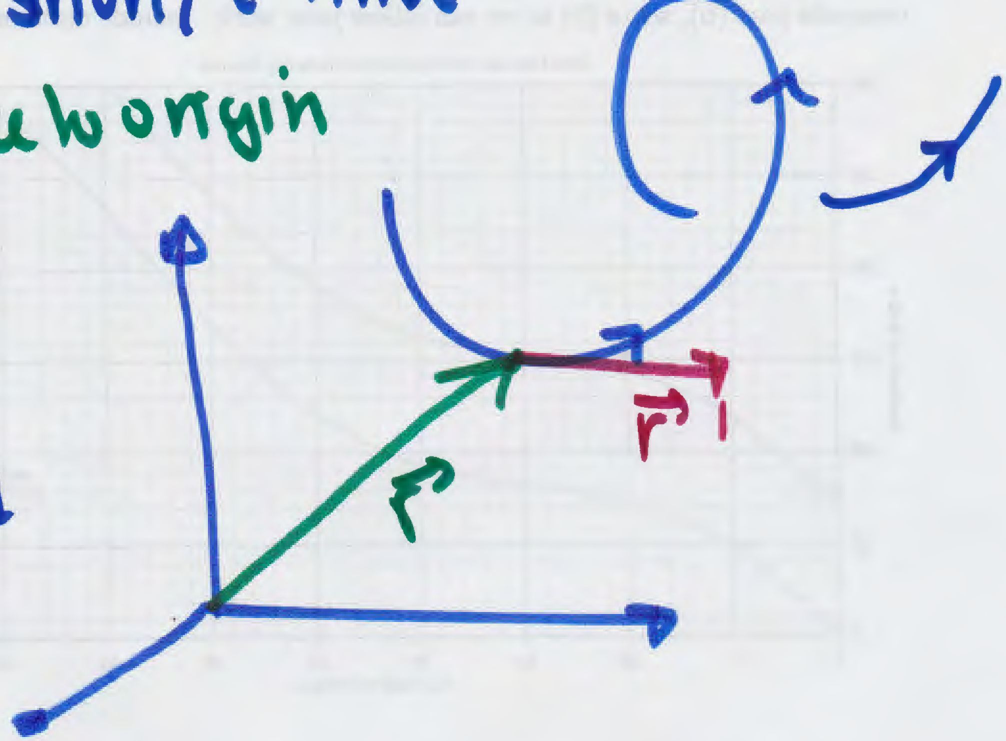
13.4 Velocity + Acceleration

$\vec{r}(t)$ position, t time

$|\vec{r}|$ - distance to origin

$\vec{r}\,'(t) = \vec{v}(t)$
velocity

$|\vec{r}\,'| =$ speed



$\vec{r}\,''(t) = \vec{v}\,'(t) = \vec{a}(t)$

ex: $\vec{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle$

$\vec{v} = \vec{r}\,' = \langle -2\sin t, 3, 2\cos t \rangle$ velocity

speed $|\vec{r}\,'| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}$

(in this example speed is constant)

$\vec{a} = \vec{v}\,' = \vec{r}\,'' = \langle -2\cos t, 0, -2\sin t \rangle$ acceleration

Differentiate

$$\vec{r} \longrightarrow \vec{v} \longrightarrow \vec{a}$$

$\longleftarrow$ integrate (remember integration constant)

ex: If $\vec{a}(t) = \langle t, 3, \sin t \rangle$, $\vec{v}(0) = \langle 1, 0, 1 \rangle$, $\vec{r}(0) = \langle 0, 4, 0 \rangle$,

find $\vec{r}(t)$.

$$\vec{v}(t) = \int \langle t, 3, \sin t \rangle \, dt$$

$$= \left\langle \frac{t^2}{2}, 3t, -\cos t \right\rangle + \vec{C}$$

$$\langle 1, 0, 1 \rangle = \vec{v}(0) = \langle 0, 0, -1 \rangle + \vec{C}$$

$$\langle 1, 0, 1 \rangle - \langle 0, 0, -1 \rangle = \vec{C}$$

$$\langle 1, 0, 2 \rangle = \vec{C}$$

$$\vec{v}(t) = \left\langle \frac{t^2}{2} + 1, 3t, -\cos t + 2 \right\rangle$$

$$\vec{r}(t) = \int \left\langle \frac{t^2}{2}+1,\ 3t,\ -\cos t+2 \right\rangle dt$$

$$= \left\langle \frac{t^3}{6}+t,\ \frac{3t^2}{2},\ -\sin t+2t \right\rangle + \vec{C}$$

$$\langle 0,4,0\rangle = \vec{r}(0) = \langle 0,0,0\rangle + \vec{C}$$

$$\vec{r}(t) = \left\langle \frac{t^3}{6}+t,\ \frac{3t^2}{2}+4,\ -\sin t+2t \right\rangle$$

—o—

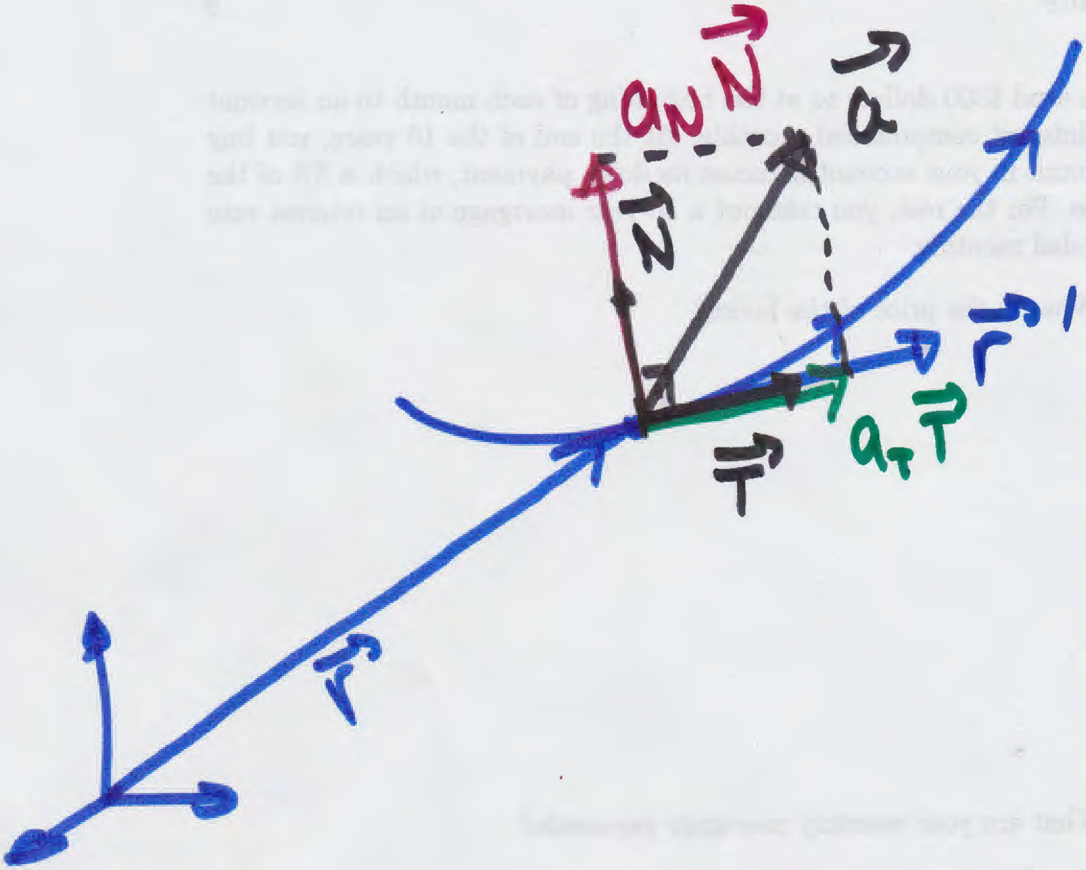
$\vec{v} = \vec{r}\,'$ can change

speed can change

direction can change

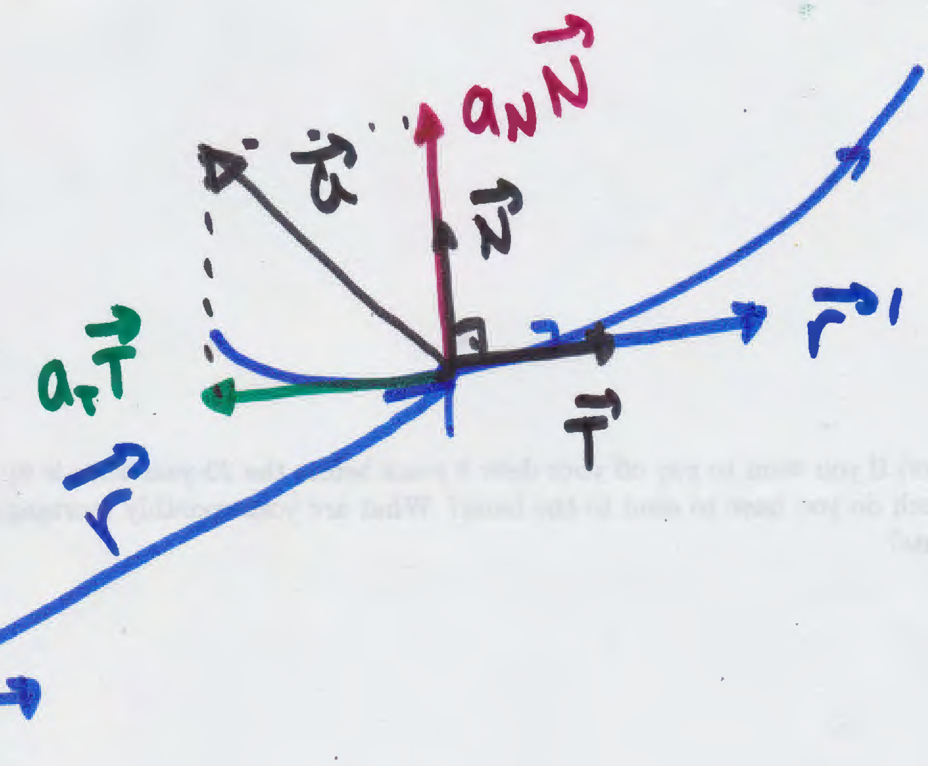
from $a_T\vec{T}$

from $a_N\vec{N}$



speed up

$$\vec{T} = \frac{1}{|r'|}\vec{r}\,'$$



$a_T < 0$

slow down

ex: linear

$\vec{r} = \langle 1+2t, 3-5t, 7+6t \rangle$

$\vec{r}\,' = \langle 2, -5, 6 \rangle$

$\vec{r}\,'' = \vec{a} = \vec{0}$

neither changes speed nor turns

ex: $\vec{r} = \langle 1+2t^2, 3-5t^2, 7+6t^2 \rangle$

traces the same line.

does <u>not</u> turn $\quad a_N \vec{N} = \vec{0}$

it speeds up

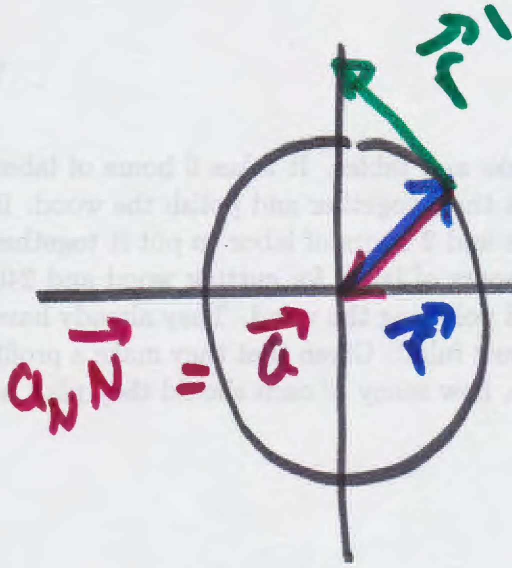
check $\vec{a} \neq \vec{0}$

ex: $\vec{r} = \langle \cos t, \sin t \rangle$ (2D)

$\vec{r}\,' = \langle -\sin t, \cos t \rangle \quad |\vec{r}\,'| = 1$ constant

so I expect $a_T \vec{T} = \vec{0}$

$\vec{r}\,'' = \langle -\cos t, -\sin t \rangle$



want: Given any $\vec r$

we want to write

$$\vec a = \underline{\underline{a_N}}\, \vec N + a_T \vec T$$

—o—

$$\vec a = \vec v\,' = (\vec r\,')'$$

$$= (|\vec r\,'|\vec T)'$$

product rule $= |\vec r\,'|'\,\vec T + |\vec r\,'|\,\vec T\,'$

$$= \underbrace{|\vec r\,'|'}_{a_T}\,\vec T + \underbrace{|\vec r\,'|\,|\vec T\,'|}_{a_N}\,\vec N$$

$$\vec N = \frac{1}{|\vec T\,'|}\vec T\,'$$

not practical for computation.

I need better formulas

First : $a_T = \boxed{|\vec{r}\,'|'}$

start : $|\vec{r}\,'|^2 = \vec{r}\,' \cdot \vec{r}\,'$

differentiate

$$\cancel{2}|\vec{r}\,'|\boxed{|\vec{r}\,'|'} = \vec{r}\,'' \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,'' = \cancel{2}(\vec{r}\,' \cdot \vec{r}\,''$$

chain

product

Formula for a_T

$$\boxed{a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|}}$$

$a_T = 0 \rightarrow$ speed not changing

$a_T > 0 \rightarrow$ speeding up

$a_T < 0 \rightarrow$ slowing down.

Now, $a_N = |\vec{r}'||\vec{T}'|$

Recall: $K = \dfrac{|\vec{T}'|}{|\vec{r}'|}$

so $a_N = K|\vec{r}'|^2$

recall: $K = \dfrac{|r' \times \vec{r}''|}{|r'|^3}$

so $\boxed{a_N = \dfrac{|\vec{r}' \times \vec{r}''|}{|r'|}}$ Formula for a_N

note $a_N \geq 0$.

Computing a_N + a_T for the spiral example, say at $t=\frac{\pi}{2}$.

$$\vec{r}\,'\left(\frac{\pi}{2}\right) = \langle -2, 3, 0 \rangle$$

$$\vec{r}\,''\left(\frac{\pi}{2}\right) = \langle 0, 0, -2 \rangle$$

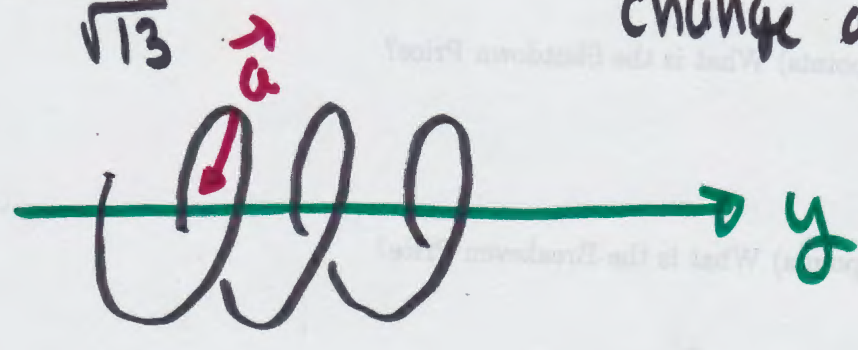
$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix} = (-6)\vec{i} - (4)\vec{j} + (0)\vec{k}$$

$$= \langle -6, -4, 0 \rangle$$

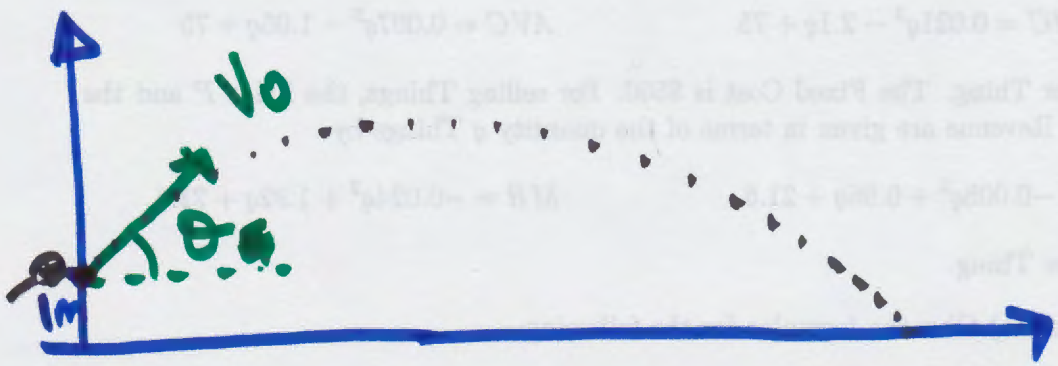
$$a_N = \frac{\sqrt{36+16+0}}{\sqrt{4+9+0}} = \frac{\sqrt{52}}{\sqrt{13}} = 2$$

$$a_T = \frac{0+0+0}{\sqrt{13}} = 0$$

speed does not change at $t=\pi/2$



example: Projectile Motion



on earth acceleration due to gravity is $9.8\,m/s^2$

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$

$$\vec{v}(0) = \langle V_0 \cos\theta, V_0 \sin\theta \rangle$$

$$\vec{r}(0) = \langle 0, 1 \rangle$$ (depends on choice of axes)

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0, -9.8 \rangle\, dt$$

$$= \langle 0, -9.8t \rangle + \vec{C}$$

$$\langle v_0\cos\theta, v_0\sin\theta\rangle = \vec{v}(0) = \langle 0, -9.8\cdot 0\rangle + \vec{C}$$

$$\vec{v}(t) = \langle v_0\cos\theta, -9.8t + v_0\sin\theta\rangle$$

$$\vec{r}(t) = \int \langle v_0\cos\theta, -9.8t + v_0\sin\theta\rangle\, dt$$

$$= \langle v_0\cos\theta\, t, -4.9t^2 + v_0\sin\theta\, t\rangle + \vec{C}$$

$$\langle 0,1\rangle = \vec{r}(0) = \langle 0,0\rangle + \vec{C}$$

$$\vec{r}(t) = \langle v_0\cos\theta\, t, -4.9t^2 + v_0\sin\theta\, t + 1\rangle$$

If you eliminate t:

$$y = -4.9\left(\frac{x}{v_0\cos\theta}\right)^2 + v_0\sin\theta\left(\frac{x}{v_0\cos\theta}\right) + 1$$

parabola equation.