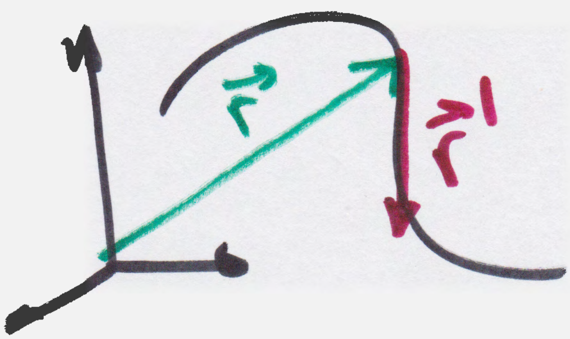
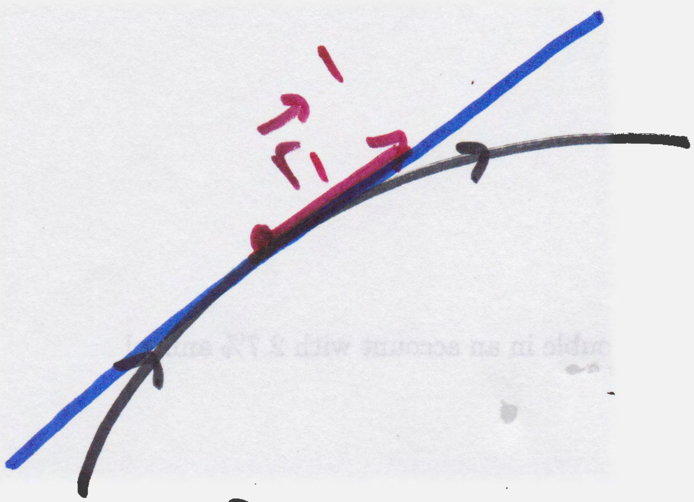
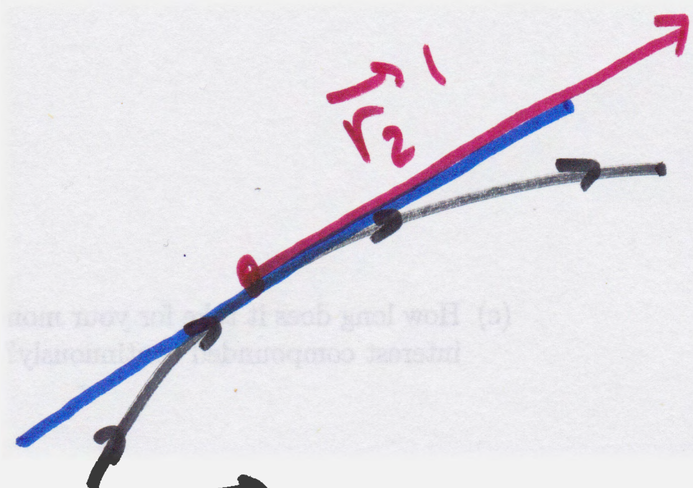
13.3 Arclength + Curvature

$\vec{r}$: position $\qquad$ $\vec{r}\,'$: velocity





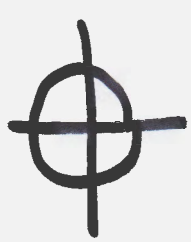
$\vec{r}_1$



$\vec{r}_2$

(stretching faster)

ex:



$\vec{r}_1(t) = \langle \cos t, \sin t \rangle$

$\vec{r}_2(t) = \langle \cos 5t, \sin 5t \rangle$

$$|\vec{r}\,'| = \text{speed}$$

$$\boxed{\begin{array}{l}\text{Distance travelled is the integral of speed.} \\[1em] \text{Arclength} = \displaystyle\int_a^b |\vec{r}\,'(t)|\,dt\end{array}}$$

ex: circumference of the unit circle

$$\int_0^{2\pi} |\vec{r}_1{}'(t)|\,dt$$

$0 \le t \le 2\pi$

<u>one</u> revolution

$$\int_0^{2\pi/5} |\vec{r}_2{}'(t)|\,dt$$

$0 \le t \le \frac{2\pi}{5}$

<u>one</u> revolution.

$\vec{r} = \langle x(t), y(t), z(t) \rangle$

$\vec{r}\,' = \langle x', y', z' \rangle$

$|\vec{r}\,'| = \sqrt{(x')^2 + (y')^2 + (z')^2}$

So I will have a square root in my integral

If I'm lucky

$|\vec{r}\,'| = \text{constant}$

or I could have a perfect square inside $\sqrt{}$

$\int \sqrt{a+bt}\, dt$ I can do u-sub

$\int \sqrt{\text{quadratic}}\, dt$ inverse trig sub

Anything worse I'm stuck.

example: Find the length of one turn of the spiral

$$\vec{r}(t) = \langle \underline{\cos t}, \underline{\sin t}, t \rangle$$

$$\vec{r}\,' = \langle -\sin t, \cos t, 1 \rangle$$

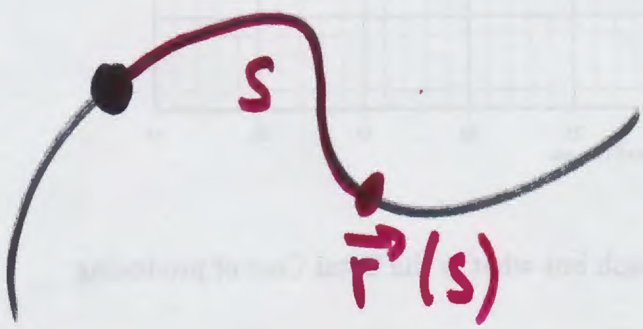
$$\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 + 1}\;dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,t \Big|_0^{2\pi} = 2\pi\sqrt{2}$$

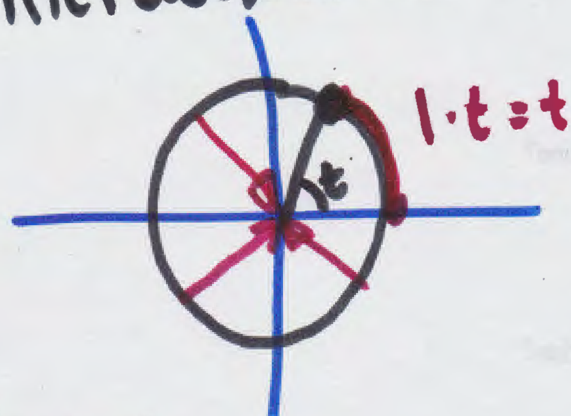
What is a good parameter for a curve?

A good parameter is arclength

We fix a point on the curve



ex: $\vec{r}(t) = \langle \cos t, \sin t \rangle$ is with respect to arclength measured counterclockwise from $(1,0)$



radius = 1

Given some $\vec{r}(t)$ and a fixed point on it, can we find ~~a~~ the parametrization with respect to arclength measured from that point?

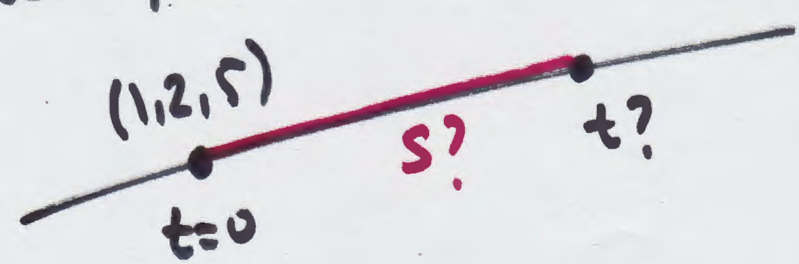
In theory, yes.

In practice, usually no. (computational issues)

ex: Reparametrize

$$\vec{r}(t) = \langle 1+t, 2-3t, 5+2t \rangle$$

with respect to arclength measured from the point with $t=0$.



STEP 1 I need to get s in terms of t.

$$s = \int_0^t |\vec{r}\,'(u)|\,du$$

switch to u for dummy variable because t is in the limit of the integral

$$\vec{r}\,' = \langle 1, -3, 2 \rangle$$

$$|\vec{r}\,'| = \sqrt{1+9+4} = \sqrt{14}$$

$$s = \int_0^t \sqrt{14}\,du = \sqrt{14}\,u \Big|_0^t = \sqrt{14}\,t$$

$$\boxed{s = \sqrt{14}\,t}$$

If $|\vec{r}\,'|$ is a nasty $\sqrt{}$, I may not be able to finish this integration and get the formula for s.

STEP 2 Solve for t in terms of s

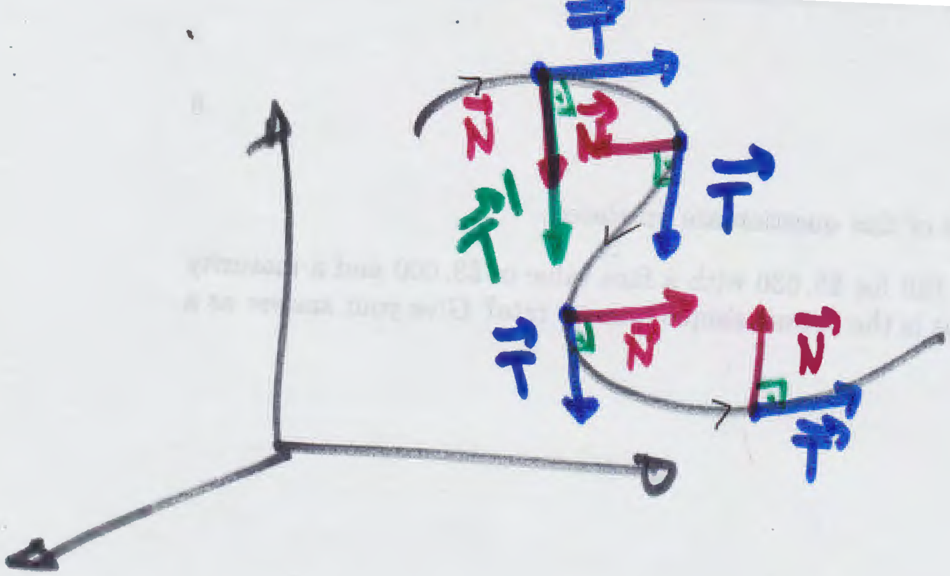
$$s = \sqrt{14}\, t$$

$$\frac{s}{\sqrt{14}} = t$$

I was lucky that the formula for t was simple.

Step 3 substitute

$$\vec{r}(s) = \left\langle 1 + \frac{s}{\sqrt{14}},\ 2 - \frac{3s}{\sqrt{14}},\ 5 + \frac{2s}{\sqrt{14}} \right\rangle$$



$$\vec{T} = \frac{1}{|\vec{r}\,'|}\,\vec{r}\,'$$

$\vec{N}$ is a unit vector which points in the direction the curve is turning.

ex:  $\vec{N} = \vec{0}$ for a linear function

it tells me how $\vec{T}$ is changing

$$\boxed{\vec{N} = \frac{1}{|\vec{T}'|}\,\vec{T}'}$$

unit normal

it is alway
<u>orthogonal</u>
to $\vec{T}$

Product Rules of Vector Differentiation

ordinary product rule: $(fg)' = f'g + fg'$

$f(t)$: function

$\vec{u}(t), \vec{v}(t)$: vector functions

ex: $f(t) = e^t$

$\vec{u}(t) = \langle t, t^2, t^3 \rangle$

$\vec{v}(t) = \langle \sin t, \cos t, \tan t \rangle$

3 products so 3 product rules

$$\left(f(t)\vec{u}(t)\right)' = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$$

$$\rightarrow \left(\vec{u} \cdot \vec{v}\right)' = \vec{u}\,' \cdot \vec{v} + \vec{u} \cdot \vec{v}\,'$$

$$\left(\vec{u} \times \vec{v}\right)' = \vec{u} \times \vec{v}\,' + \vec{u}\,' \times \vec{v}$$

ex: check at least one by computing both sides.

To show $\vec{T} \perp \vec{N}$

we need to show $\vec{T} \perp \vec{T}'$

i.e. we need $\vec{T} \cdot \vec{T}' = 0$.

we start : $|\vec{T}| = 1$

so $\vec{T} \cdot \vec{T} = 1$

differentiate both sides:

product rule

$$\vec{T}' \cdot \vec{T} + \vec{T} \cdot \vec{T}' = 0$$

$$2(\vec{T} \cdot \vec{T}') = 0$$

$$\vec{T} \cdot \vec{T}' = 0$$

so $\vec{T} \perp \vec{T}'$

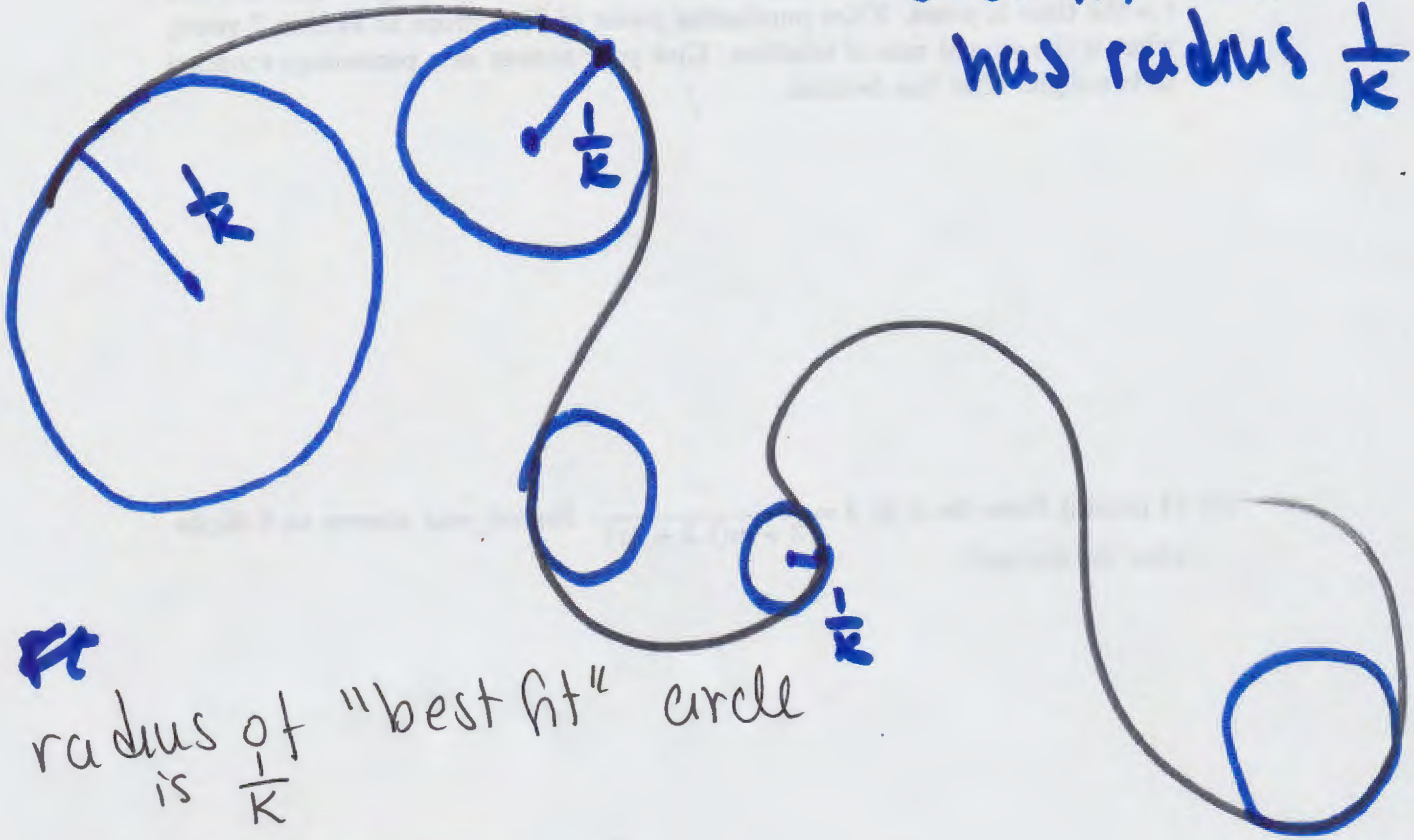
and then

$\vec{T} \perp \vec{N}$

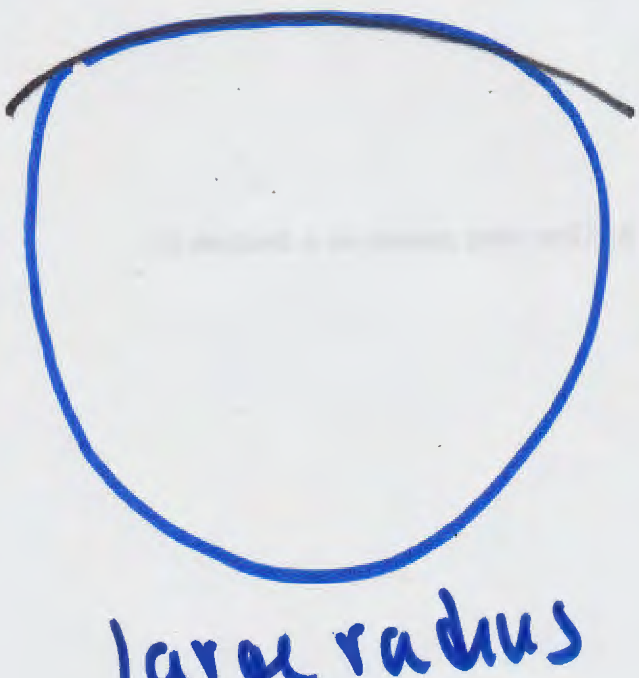
Curvature κ (kappa)

measures how the curve bends

"best fit" circle has radius $\frac{1}{\kappa}$



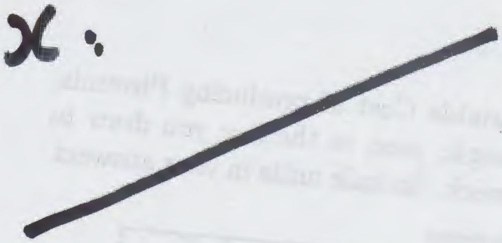
radius of "best fit" circle is $\frac{1}{\kappa}$



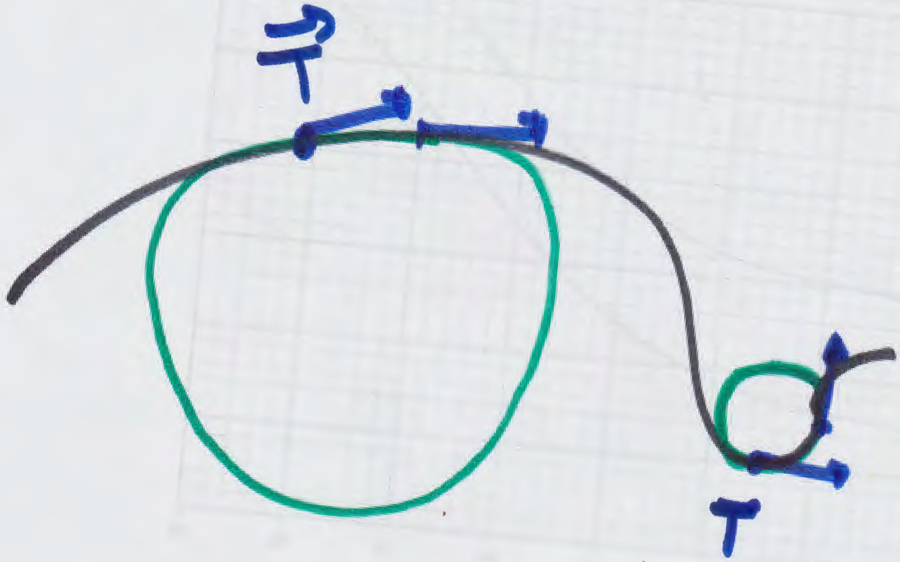
large radius small κ

small radius large κ.

ex.:



$\kappa = 0.$



want: how much $\vec{T}$ is changing with respect to <u>arclength</u>

$$\boxed{\kappa = \left| \frac{d\vec{T}}{ds} \right| \qquad \text{curvature definition}}$$

In practice, it's mostly not possible to get $\vec{T}$ in terms of s.
Because reparametrization may fail in step 1 or step 2

$$K = \frac{|\vec{T}'|}{|\vec{r}'|}$$

First formula for curvature

~~no~~ unless ~~the~~ we have already computed $\vec{T}$ for other part of a question, we don't use this.

we use

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

Second Formula for curvature