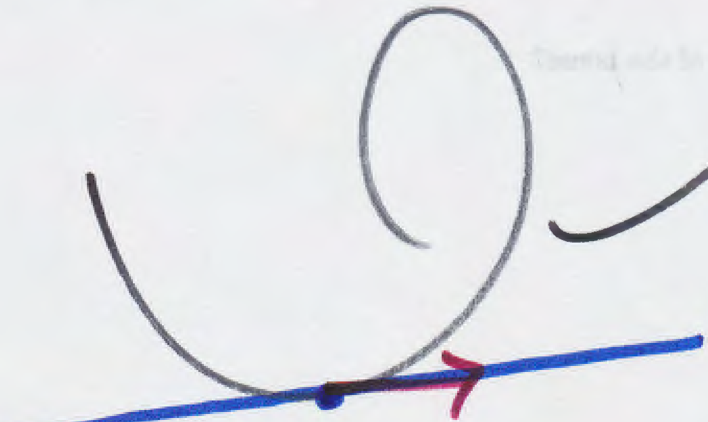
13.2 Derivatives + Integrals of Vector Functions

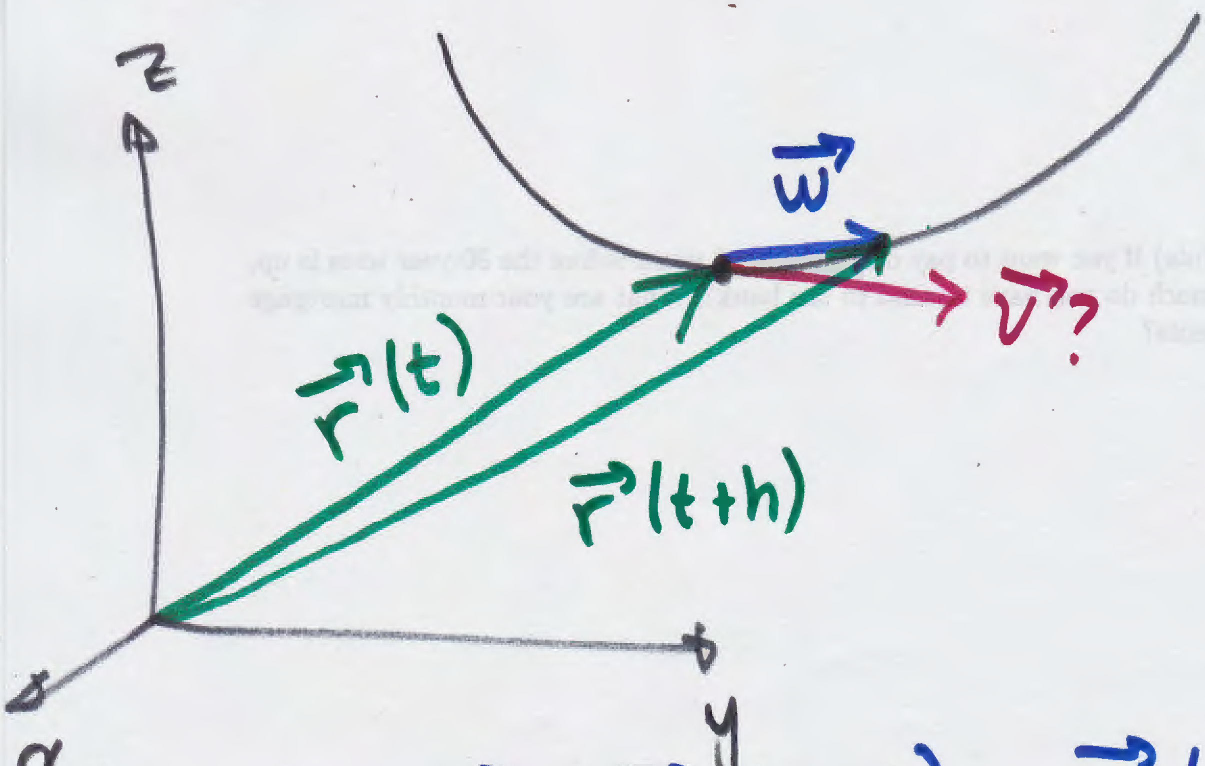
$\vec{r}$



tangent line?

need: point (given)

<u>direction</u> ?



$$\vec{w} = \vec{r}(t+h) - \vec{r}(t)$$

$\vec{w}$ + $\vec{v}$ are not really parallel

if I make h very small

$\vec{r} = \langle x(t), y(t), z(t) \rangle$

then $\vec{w}$ + $\vec{v}$ are almost parallel.

want to take $h \to 0$

☹ when $h \to 0$, $\vec{w} \to \vec{0}$

$\vec{0}$ has no direction.

To avoid $\vec{w}$ ~~shrng~~ shrinking in size I multiply it by a <u>large</u> scalar: $\frac{1}{h}$

$$\lim_{h\to 0} \frac{1}{h}\left(\vec{r}(t+h)-\vec{r}(t)\right) = \vec{r}\,'(t)$$

$\vec{v}$ $\quad \frac{1}{h}\left(\vec{r}(t+h)-\vec{r}(t)\right)$

Done with geometry; execute idea:

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{1}{h}\left(\vec{r}(t+h)-\vec{r}(t)\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\langle x(t+h), y(t+h), z(t+h)\rangle - \langle x(t), y(t), z(t)\rangle\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\langle x(t+h)-x(t), y(t+h)-y(t), z(t+h)-z(t)\rangle$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h}\right\rangle$$

you take limits of vector functions componentwise

$$\left\langle \lim_{h\to 0}\frac{x(t+h)-x(t)}{h}, \lim_{h\to 0}\frac{y(t+h)-y(t)}{h}, \lim_{h\to 0}\frac{z(t+h)-z(t)}{t} \right\rangle$$

$$\overset{\text{calc I}}{=} \left\langle x'(t), y'(t), z'(t) \right\rangle$$

So: you differentiate vector functions componentwise.

! $\vec{r}\,'(t)$ is parallel to the tangent line.

example: Let $\vec{r}(t) = \langle t^2+5t+2, e^{3t}+4, \sin(t^2) \rangle$

Find the (vector) equation of the tangent **line** at $t = \underline{0}$

* point / initial position $\qquad \vec{r}(\underline{0})$

* direction $\qquad \vec{r}\,'(\underline{0})$

$$\vec{r}(0) = \langle 2, 5, 0 \rangle$$

$$\vec{r}\,' = \langle 2t+5, 3e^{3t}, 2t\cos(t^2) \rangle$$

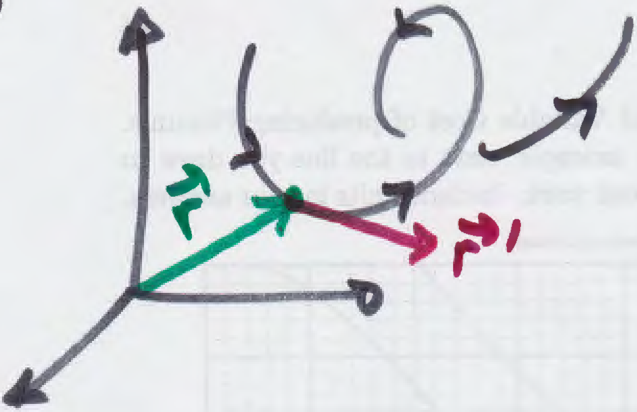
$$\vec{r}\,'(0) = \langle 5, 3, 0 \rangle$$

$$\vec{r}_\ell(t) = \langle 2, 5, 0 \rangle + t \langle 5, 3, 0 \rangle$$
$$= \langle 2+5t, 5+3t, 0 \rangle$$

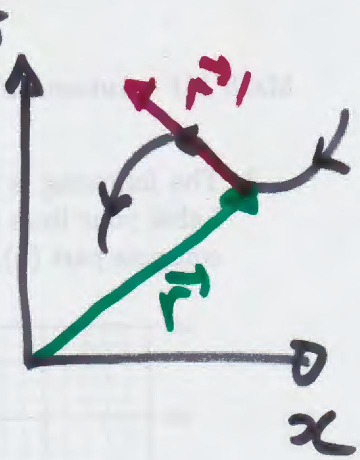
Parametric: $x = 2+5t \quad y = 5+3t \quad z = 0$

Symmetric: $\dfrac{x-2}{5} = \dfrac{y-5}{3}, \quad z = 0$

3D



2D



$\vec{r}$ is always sketched / visualized starting at the origin + pointing at the current position.

$\vec{r}'$ is always sketched / visualized starting at the point of tangency

—o—

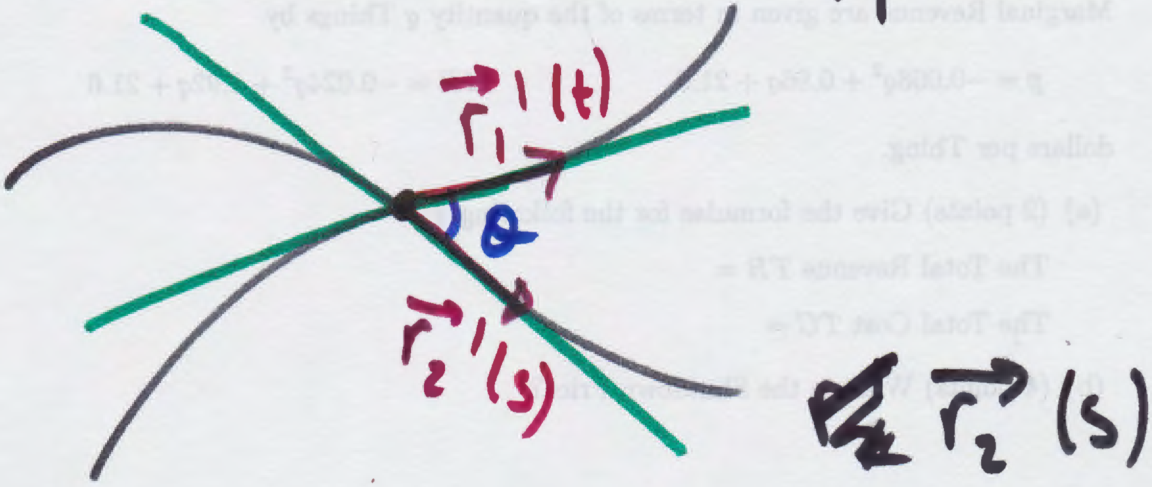
We can find the angle of intersection of two curves (if they intersect)

To find intersection of $\vec{r}_1$ + $\vec{r}_2$

$$\vec{r}_1(t) = \vec{r}_2(s)$$

different

If you find a pair of numbers t & s satisfying the equation $\vec{r}_1(t)$



θ will be the angle between $\vec{r}_1\,'(t)$ + $\vec{r}_2\,'(s)$

we can

$$\vec{r} \xrightarrow[\text{(componentwise)}]{\text{differentiate}} \vec{r}\,'$$

so we can go backwards

$$\vec{r}\,' \xrightarrow[\text{(componentwise)}]{\text{integrate}} \vec{r}$$

Don't forget $+\vec{C}$ in your (vector) integrals.

example: Find $\vec{r}(t)$ if

$$\vec{r}\,'(t) = \langle \cos t,\ e^{t/2},\ t^2+5t+3 \rangle$$

and $\vec{r}(0) = \langle 1, 5, 3 \rangle$.

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$$

$$= \int \langle \cos t, e^{t/2}, t^2 + 5t + 3 \rangle \, dt$$

$$= \left\langle \sin t, 2e^{t/2}, \frac{t^3}{3} + \frac{5t^2}{2} + 3t \right\rangle + \vec{C}$$

Some people prefer to do:

$$\int \langle \cos t, e^{t/2}, t^2 + 5t + 3 \rangle \, dt$$

$$= \left\langle \cancel{\cos} \sin t + C_1, 2e^{t/2} + C_2, \frac{t^3}{3} + \frac{5t^2}{2} + 3t + C_3 \right\rangle$$

Do ~~this~~ what you prefer

I'll use $\vec{r}(0) = \langle 1, 5, 3 \rangle$:

$$\langle 1, 5, 3 \rangle = \vec{r}(0) = \langle 0, 2, 0 \rangle + \vec{C}$$

$$\langle 1, 5, 3 \rangle - \langle 0, 2, 0 \rangle = \vec{C}$$

$$\vec{C} = \langle 1, 3, 3 \rangle$$

So:

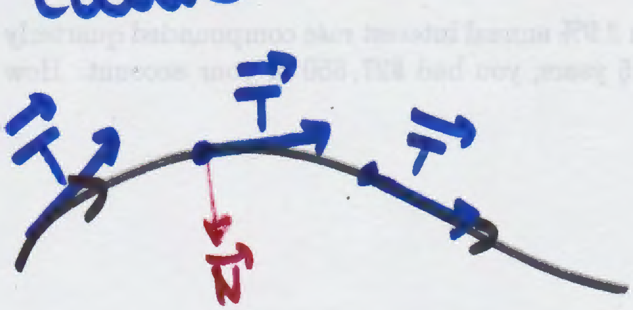
$$\vec{r}(t) = \left\langle \sin t, 2e^{t/2}, \frac{t^3}{3} + \frac{5t^2}{2} + 3t \right\rangle + \langle 1, 3, 3 \rangle$$

$$= \left\langle \sin t + 1, 2e^{t/2} + 3, \frac{t^3}{3} + \frac{5t^2}{2} + 3t + 3 \right\rangle.$$

The unit vector in the direction of $\vec{r}\,'$ is

$$\vec{T} = \frac{1}{|\vec{r}\,'|}\,\vec{r}\,'$$

is called the unit tangent vector



ex: Compute the unit tangent vector to $\vec{r}(t) = \langle 4\sin(3t), 5e^{2t} - 1, t^3 \rangle$ at the point where $t = \pi$.

$$\vec{r}\,' = \langle 3\cos(3t), 10e^{2t}, 3t^2 \rangle$$

$$\vec{r}\,'(\pi) = \langle -3, 10e^{2\pi}, 3\pi^2 \rangle$$

$$\vec{T} = \frac{1}{\sqrt{9 + 100e^{4\pi} + 9\pi^4}} \langle -3, 10e^{2\pi}, 3\pi^2 \rangle$$

$$= \left\langle \frac{-3}{\sqrt{9+100e^{4\pi}+9\pi^4}}, \frac{10e^{2\pi}}{\sqrt{9+100e^{4\pi}+9\pi^4}}, \frac{3\pi^2}{\sqrt{9+100e^{4\pi}+9\pi^4}} \right\rangle$$