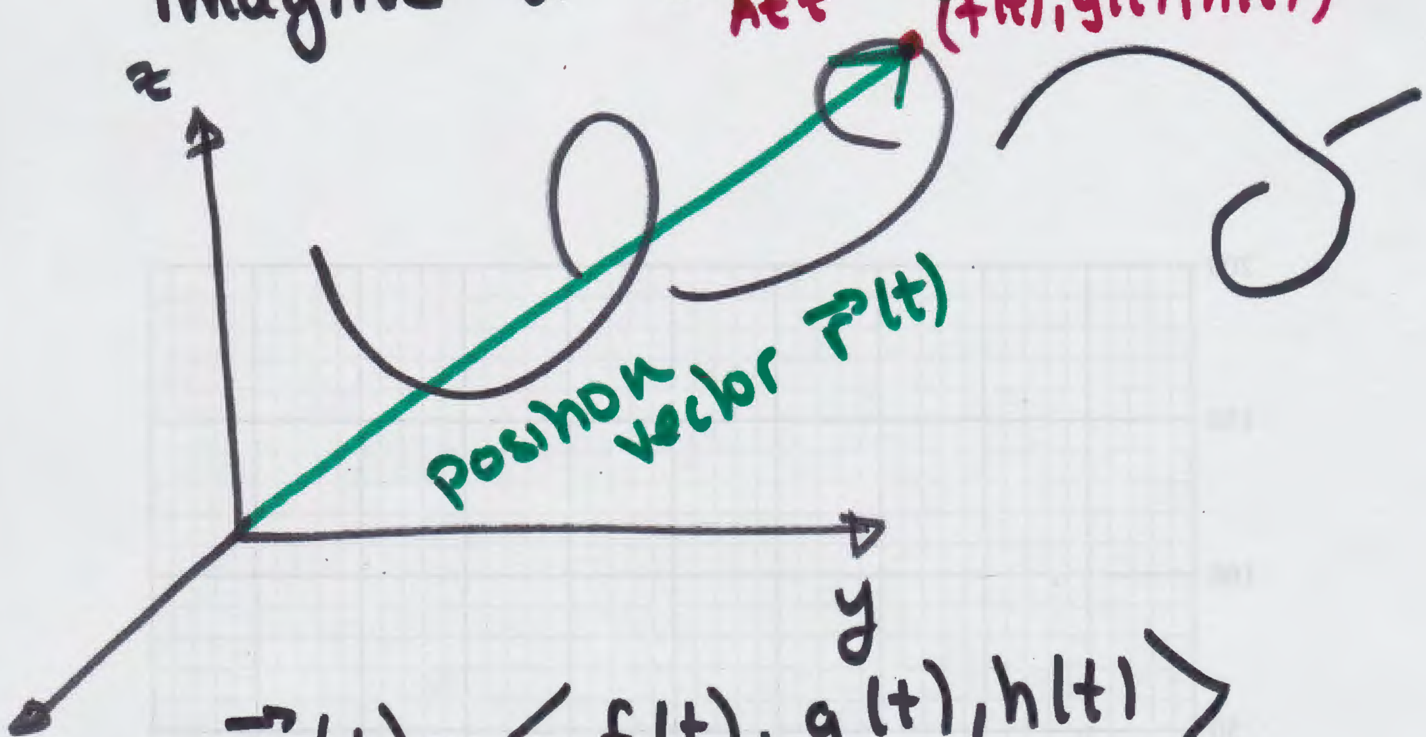
CHAPTER 13 - CALCULUS ON CURVES / VECTOR FUNCTIONS

13.1 Space Curves

space curve

$x = f(t) \qquad y = g(t) \qquad z = h(t)$

imagine a Blue Angel Plane



x

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

is a <u>vector</u> function

$\xrightarrow{t} \boxed{} \; \vec{r}(t)$

(number) (vector)

ex: A <u>line</u> is a straight curve

$$\vec{r}(t) = \langle 1-2t, 3+5t, 7t \rangle$$

(this used point $(1,3,0)$
and direction vector $\langle -2, 5, 7 \rangle$)

if you want, the parametric equations
$x = 1-2t \quad y = 3+5t \quad z = 7t$

ex: $x = t$ $\quad \boxed{y = \cos t \qquad z = \sin t}$

(in vector form $\vec{r}(t) = \langle t, \cos t, \sin t \rangle$)

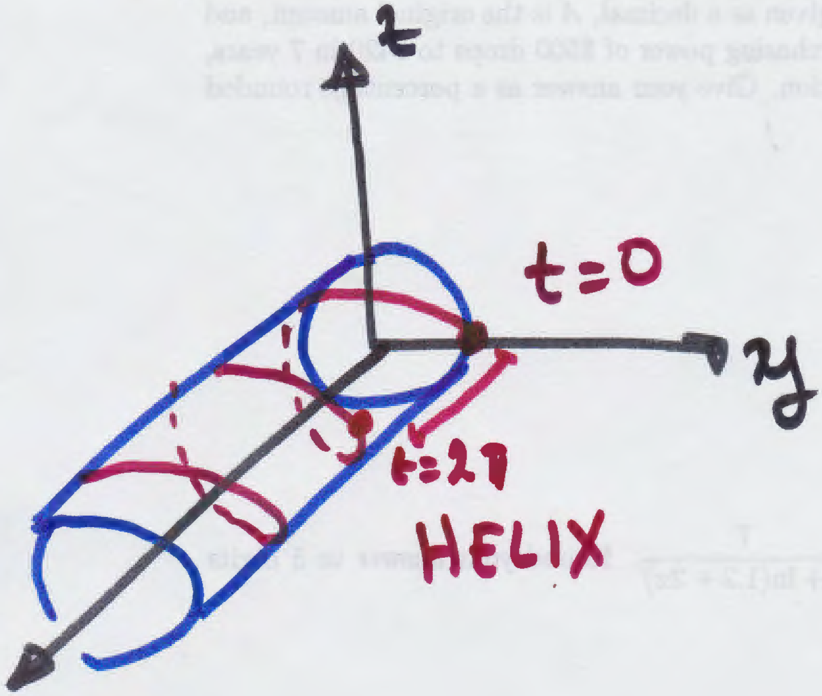
$$\boxed{\cos^2 t + \sin^2 t = 1}$$

"eliminate t"
to get a surface
equation

$$y^2 + z^2 = 1$$

For <u>any</u> t, the
curve satisfies the
equation $y^2 + z^2 = 1$
i.e. the curve is
<u>on</u> the surface
$y^2 + z^2 = 1$

$y^2+z^2=1$ is a (generalized) cylinder because x is missing



actual cylinder

Now, consider $x=t$:

As t increases, so will x.

As t increases you will go around the circle.

ex : $\vec{r}(t) = \langle \cos t \cos 5t , \cos t \sin 5t , \sin t \rangle$

SAME

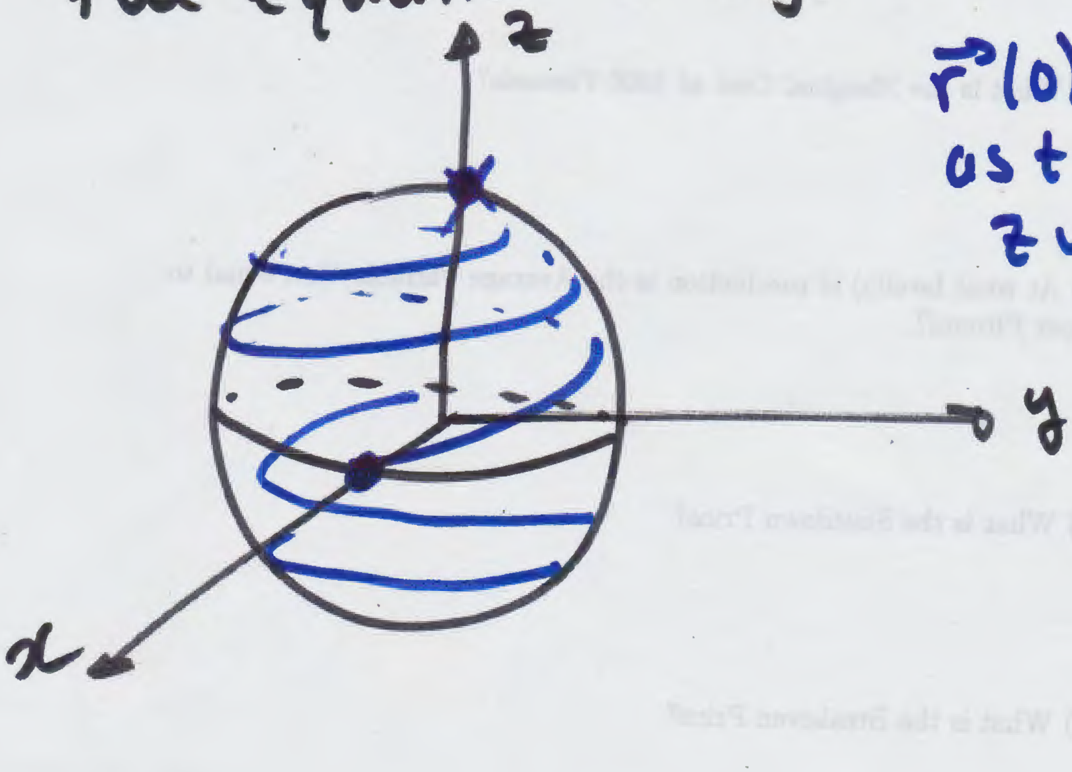
$$x^2 + y^2 = \cos^2 t \cos^2 5t + \cos^2 t \sin^2 5t$$

$$= \cos^2 t \left(\cos^2 5t + \sin^2 5t \right)$$

$$= \cos^2 t$$

$$x^2 + y^2 + z^2 = 1$$

So the curve $\vec{r}(t)$ traces is <u>on</u> the sphere because it satisfies the equation $x^2 + y^2 + z^2 = 1$ for <u>every</u> t.



$\vec{r}(0) = \langle 1, 0, 0 \rangle$

as t increases

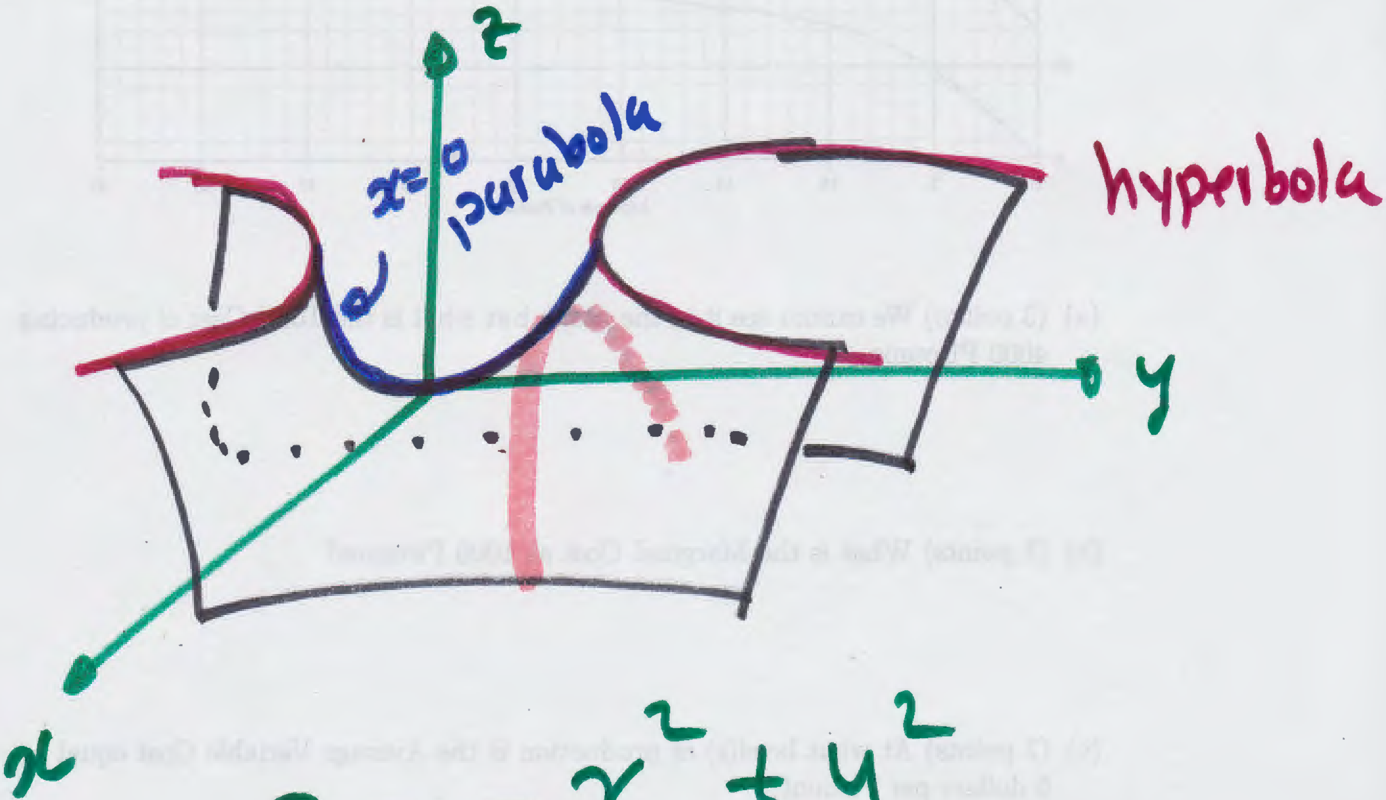
z will increase

$$(x-1)^2 + \left(\frac{y+3}{4}\right)^2 + z^2 = 1$$

is also an ellipsoid.

Its center at $(1,-3,0)$

HYPERBOLIC PARABOLOID (SADDLE)



$$z = -x^2 + y^2$$

I can sketch: $\frac{z}{5} = -\frac{x^2}{4} + y^2$

I can shift $z - 1 = -x^2 + (y+3)^2$

NOTE: A curve is <u>on a surface</u> if the components of $\vec{r}(t)$ satisfy the surface equation for <u>all</u> <u>t</u>.

example: The line $\vec{r}(t) = \langle 2+t, t, 4+t \rangle$ sits on the plane $2x+3y-5z=-16$

check: $2(2+t)+3(t)-5(4+t)$

$= 4+2t+3t-20\cancel{t}-5t = -16$ ✓

A curve <u>intersects</u> a surface at one (or more) points if the equation you get by putting the components into the surface equation has one (or more) solutions in t.

ex: The line $\vec{r}(t) = \langle t, 1+2t, 2+3t \rangle$ intersects the sphere

$$x^2+y^2+z^2 = 9$$

at :

$$t^2 + (1+2t)^2 + (2+3t)^2 = 9$$

$$\underline{t^2} + 1 + 4t + \underline{4t^2} + 4 + 12t + \underline{9t^2} = 9$$

$$14t^2 + 16t - 4 = 0$$

QF: $t = \dfrac{-16 \pm \sqrt{256 - 4(14)(-4)}}{28}$

$$\approx 0.211, -1.354$$

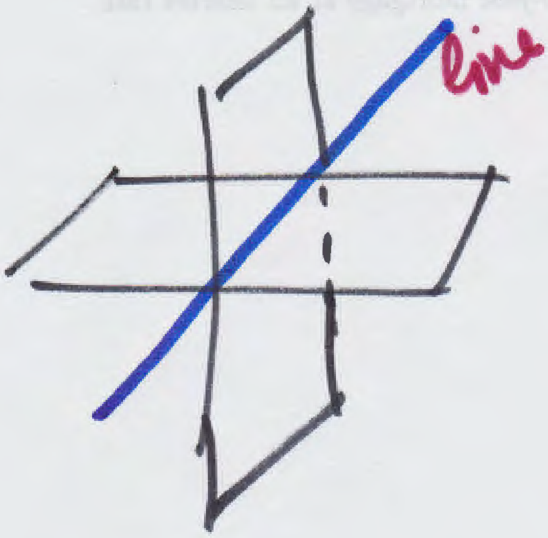
at the points

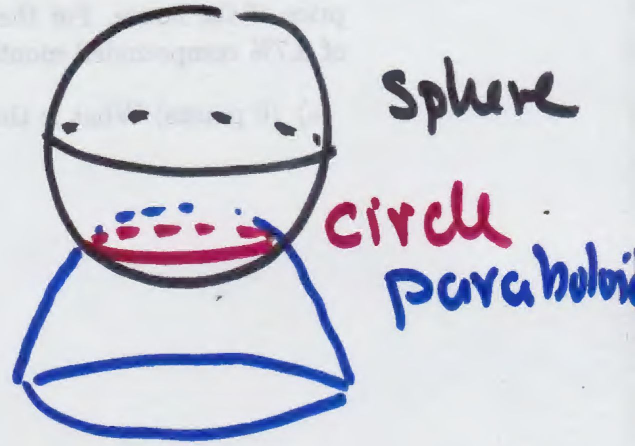
$(0.211, 1+2(0.211), 2+3(0.211))$

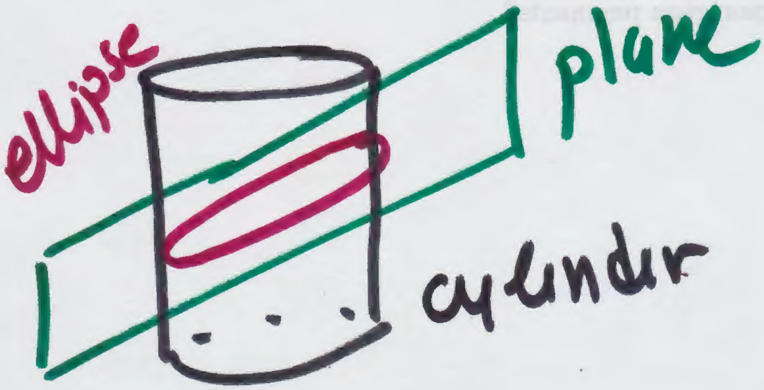
and

$(-1.354, 1+2(-1.354), 2+3(-1.354))$

The intersection of two surfaces is a curve.







Describing a curve as the intersection of two surfaces is convinient for visualization, but not suitable for calculus.

For that I need <u>vector functions</u> (or parametrization)

ex: Find a parametrization of the curve of intersection of

START HERE $\boxed{x^2+y^2=9}$ and $2x+3y-z=4$

CYLINDER

PLANE

my curve must be an ellipse.

need: $x=\underline{f(t)}$ $y=\underline{g(t)}$ $z=\underline{h(t)}$

so that they satisfy BOTH surface equations (for <u>all</u> t).

$$\boxed{9\cos^2 t + 9\sin^2 t = 9}$$

$x = 3\cos t \quad y = 3\sin t$

(others that work: $x = 3\sin t \quad y = 3\cos t$

$x = 3\sin(7t) \quad y = -3\cos(7t)$

$\vdots$

I use $2x + 3y - z = 4$

$2(3\cos t) + 3(3\sin t) - z = 4$

$6\cos t + 9\sin t - 4 = z$

one possible answer

$x = 3\cos t \quad y = 3\sin t \quad z = 6\cos t + 9\sin t - 4$

example: Parametrize the intersection of

$z = x + y^2$ $\qquad$ $z = 8 + y$

start $y = t$

then $z = 8 + t$

$8 + t = x + t^2$

$8 + t - t^2 = x$

$x = -t^2 + t + 8 \qquad y = t \qquad z = 8 + t$

<u>OR</u>

$\vec{r}(t) \langle -t^2 + t, 8, t, 8 + t \rangle$