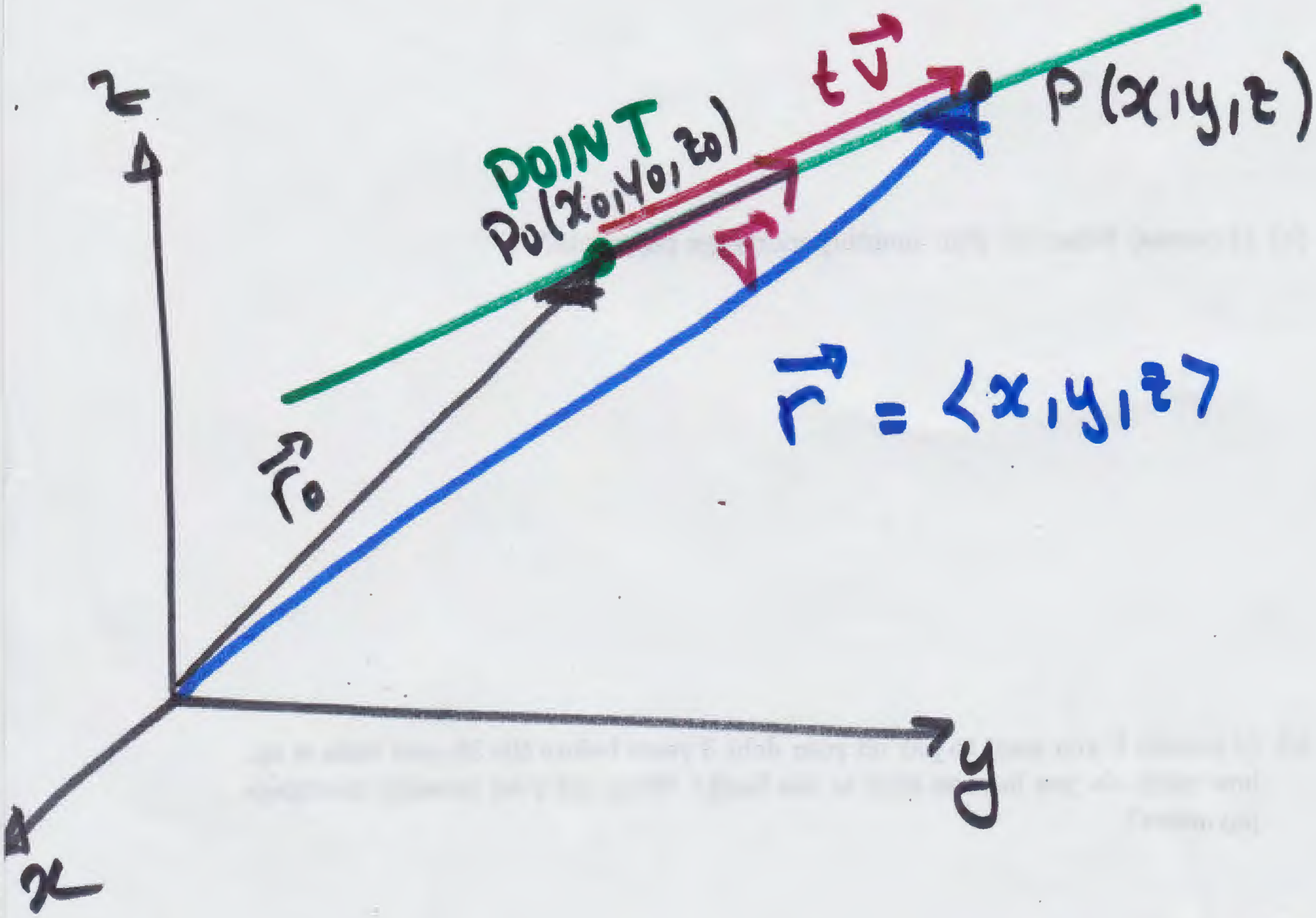
12.5 Lines and Planes in Space

LINES in space

need: POINT $P_0(x_0, y_0, z_0)$
DIRECTION VECTOR $\vec{v} = \langle a, b, c \rangle$



$\vec{r}$ points to the point on the line.

need: a parameter: t

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$

VECTOR EQUATION OF A LINE

so $\vec{r} = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

example: Write the equation of the line through $(1,7,2)$ with direction vector $\langle 0, 5, 13 \rangle$:

$$\boxed{\vec{r} = \langle 1, 7, 2 \rangle + t \langle 0, 5, 13 \rangle}$$

VECTOR equation for a line

$$\boxed{\vec{r} = \langle 1, 7 + 5t, 2 + 13t \rangle}$$

Note: In this ~~cr~~ course

$$\vec{r} = \langle x, y, z \rangle$$

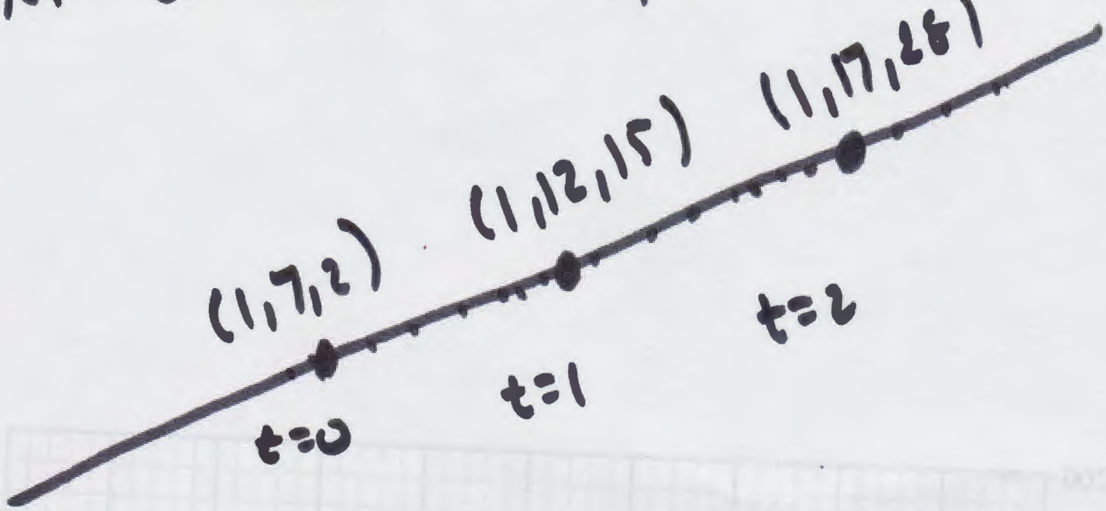
is the position vector. It points to the point (x, y, z).

I can eliminate vector notation:

$$\boxed{x = 1 \quad y = 7 + 5t \quad z = 2 + 13t}$$

$\boxed{\text{are } \underline{\text{parametric equations}}}$

for a line in space.



I can try to eliminate t:

example: Given $x = 1 - 2t \quad y = 3 + 5t \quad z = 7 - 3t$

OR $\vec{r} = \langle 1 - 2t, 3 + 5t, 7 - 3t \rangle$

$$\frac{x - 1}{-2} = t \qquad \frac{y - 3}{5} = t \qquad \frac{z - 7}{-3} = t$$

$$\frac{x-1}{-2} = \frac{y-3}{5} = \frac{z-7}{-3}$$

symmetric equations of the line.

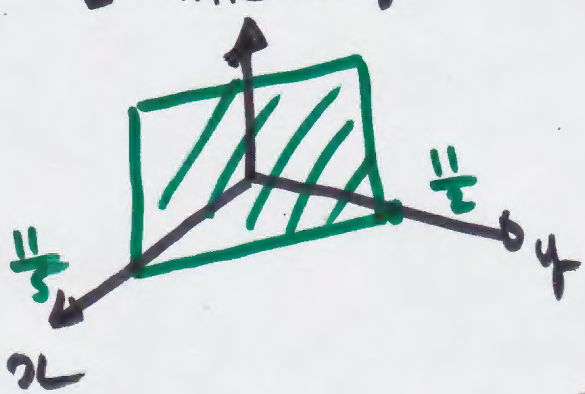
This is really 3 equations:

$$\frac{x-1}{-2} = \frac{y-3}{5} \qquad \frac{y-3}{5} = \frac{z-7}{-3} \qquad \frac{x-1}{-2} = \frac{z-7}{-3}$$

$5x-5 = -2y+6 \qquad -3y+9 = 5z-35$

$5x+2y = 11$

z missing



All 3 are planes. Their intersection is the line.

ex: $x = 1 \quad y = 7 + \underline{5}t \quad z = 2 + \underline{13}t$

symmetric equations:

$$x = 1 \qquad \frac{y-7}{5} = \frac{z-2}{13}$$

For one line, you can write many equations:

For $t = -1$ I have the point:

$$(1, 7-5, 2-13) = (1, 2, -11)$$

I can use instead of $\langle 0, 5, 13 \rangle$ the vector $3\langle 0, 5, 13 \rangle = \langle 3, 15, 39 \rangle$

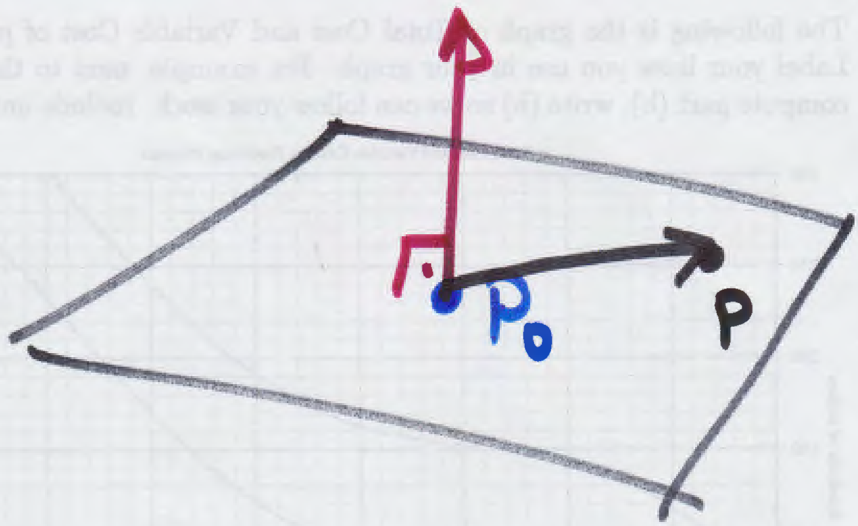
So:

$$\vec{r} = \langle 1, 2, -11 \rangle + t\langle 3, 15, 39 \rangle$$

describes the <u>same</u> line.

$x = 1 + \cancel{3t} \quad y = 2 + 15t \quad z = -11 + 34t$

PLANES in Space



I need a point $P_0(x_0, y_0, z_0)$

I need a normal (perpendicular) vector to the plane $\vec{n} = \langle a, b, c \rangle$

Given P_0, $\vec{n}$ we have a unique plane.

For any point on my plane $P(x, y, z)$

The vector

$\overrightarrow{P_0P}$ will be orthogonal (perpendic

to the vector $\vec{n}$.

$$\overrightarrow{P_0P} \bullet \vec{n} = 0$$

$$\langle x-x_0, y-y_0, z-z_0 \rangle \bullet \langle a,b,c \rangle = 0$$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

$$ax+by+cz = ax_0+by_0+cz_0$$

example: The equation of the plane through the point $(5,-1,3)$ and normal vector $\langle 4,3,2 \rangle$ is

$$4(x-5)+3(y-(-1))+2(z-3)=0$$

simplifies to

$$4x+3y+2z = 20-3+6$$

$$\boxed{4x+3y+2z = 23}$$

A plane equation in space is of the form

$$\underline{A}x+\underline{B}y+\underline{C}z = D$$

Any linear equation with x, y and z describes a plane in space.

~~The no~~ A normal is given by

$$\vec{n} = \langle A, B, C\rangle$$

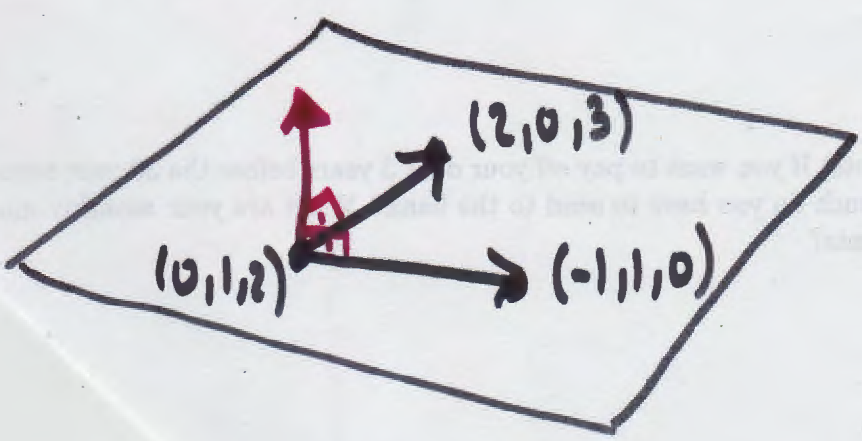
~~example~~:

3 non-linear points in space determine a unique plane.

example: Find the equation of the plane containing the points $(0,1,2)$, $(2,0,+3)$ and $(-1,1,0)$.

need: Point $(0,1,2)$

normal $\vec{n}$



$$\vec{n} = \langle 2-0, 0-1, 3-2 \rangle \times \langle -1-0, 1-1, 0-2 \rangle$$

$$= \langle 2, -1, 1 \rangle \times \langle -1, 0, -2 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ -1 & 0 & -2 \end{vmatrix} = \cancel{(2-1)\vec{i} - (-4-(-1))\vec{j} + (2-(-1))\vec{k}}$$

$$\cancel{\langle 2, 3, 1 \rangle}$$

$$= (2-0)\vec{i} - (-4-(-1))\vec{j} + (0-1)\vec{k}$$

$$= \langle 2, 3, -1 \rangle$$

$$2(x-0) + 3(y-1) - 1(z-2) = 0$$

$$2x + 3y - z = 3 - 2$$

$$\boxed{2x + 3y - z = 1}$$