12.3 The Dot Product

(also called scalar product because the answer is a scalar (number))

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \cdot \vec{b} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{scalar (number)}}$$

vectors

ex: $\langle 1, -2, 0 \rangle \cdot \langle 3, 5, 7 \rangle$

$= 1(3) + (-2)5 + 0(7) = -7$

nice properties (we take these for granted)

1) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2) $\vec{a} \cdot \vec{0} = 0$

vector

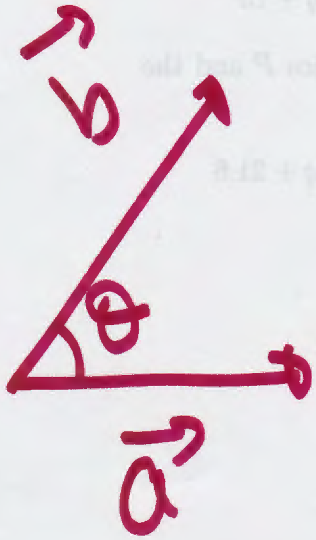
number

3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$\langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle$
$= a_1^2 + a_2^2 + a_3^2$

Theorem (what makes the dot product useful)



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

ex: Find the angle between $\langle 6,-3,2\rangle$ and $\langle 2,1,-2\rangle$.

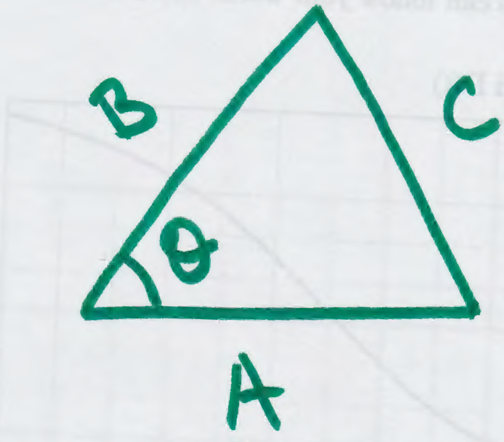
$$12 - 3 - 4 = \sqrt{36+9+4}\,\sqrt{4+1+4}\,\cos\theta$$

$$\frac{5}{21} = \frac{5}{\sqrt{49}\,\sqrt{9}} = \cos\theta$$

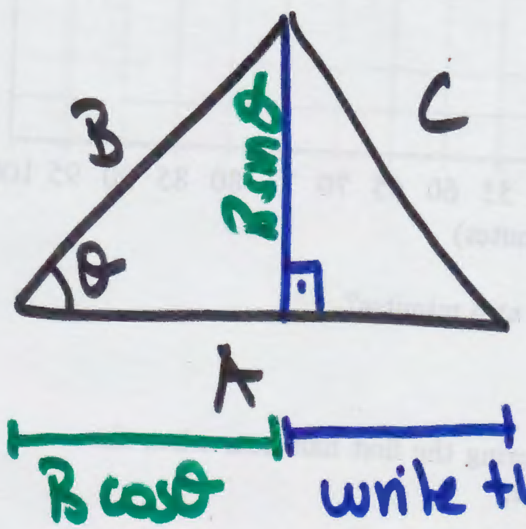
$$\theta = \cos^{-1}\left(\frac{5}{21}\right)$$

since $\cos\theta > 0$, this an acute angle.

Recall: Law of Cosines



$$A^2+B^2-2AB\cos\theta = C^2$$



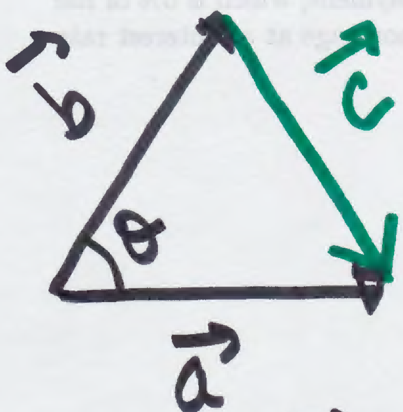
write this length in two ways

Pyth. Thm.

$$A - B\cos\theta = \sqrt{C^2-(B\sin\theta)^2}$$

square both sides + clean up to get the answer.

Proof of $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$



$$\vec{b}+\vec{c}=\vec{a}$$
$$\vec{c}=\vec{a}-\vec{b}$$

Law of Cosines:

$$|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\theta=|\vec{a}-\vec{b}|^2$$
$$\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2|\vec{a}||\vec{b}|\cos\theta=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$
$$\cancel{\vec{a}\cdot\vec{a}}+\cancel{\vec{b}\cdot\vec{b}}-2|\vec{a}||\vec{b}|\cos\theta=\cancel{\vec{a}\cdot\vec{a}}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\cancel{\vec{b}\cdot\vec{b}}$$
$$-\cancel{2}|\vec{a}||\vec{b}|\cos\theta=-\cancel{2}(\vec{a}\cdot\vec{b})$$

example: Check that $\langle 2,5,1\rangle$ and $\langle -3,1,1\rangle$ are orthogonal.

perpendicular

$$\langle 2,5,1\rangle \cdot \langle -3,1,1\rangle = -6+5+1=0$$

$\cos\frac{\pi}{2}=0$ therefore the vectors are ~~perp~~ orthogonal.

— o —

How do you check two vectors are parallel?

×1

✗ $\langle 1,2,-7\rangle \quad \langle 4,2,3\rangle$

×4

✓ $-2\langle 2,0,-5\rangle = \langle -4,0,10\rangle$

×3

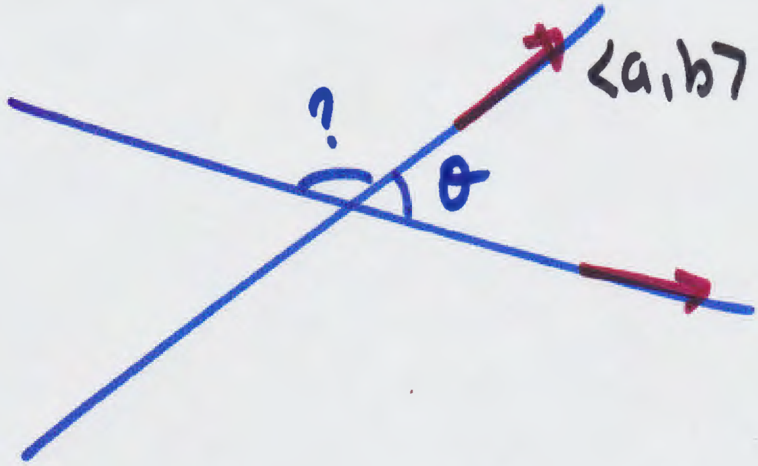
✗ $\langle 2,3,-7\rangle \quad \langle 6,9,15\rangle$

×3

$-7(3)=-21$

If one is a <u>scalar</u> multiple of the other.

example: Find the acute angle between the lines $y=2x+3$ and $2x+5y=7$ (on the plane).



$\langle a,b\rangle$ then $\frac{b}{a}$ = slope

$y=2x+3$, slope $=2$ so $\vec{a}=\langle 1,2\rangle$

$$2x+5y=7$$
$$5y=-2x+7$$
$$y=-\frac{2}{5}x+\frac{7}{5}$$

slope $=-\frac{2}{5}$ $\quad \vec{b}=\langle 1,-\frac{2}{5}\rangle$

fraction

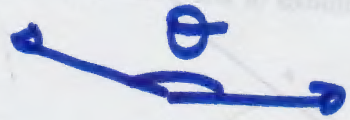
use $\vec{b}=\langle 5,-2\rangle$ instead

(I can also use $\langle -5,2\rangle$)

I'll find the angle between $\langle 1,2 \rangle$ and $\langle -5,2 \rangle$

$\cos\theta > 0$

$\theta = \ldots$



$\cos\theta < 0$

θ obtuse

I need to modify my answer to answer the question

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$-5+4 = \sqrt{1+4}\ \sqrt{25+4}\ \cos\theta$$

$$\frac{-1}{\sqrt{5}\sqrt{29}} = \cos\theta$$

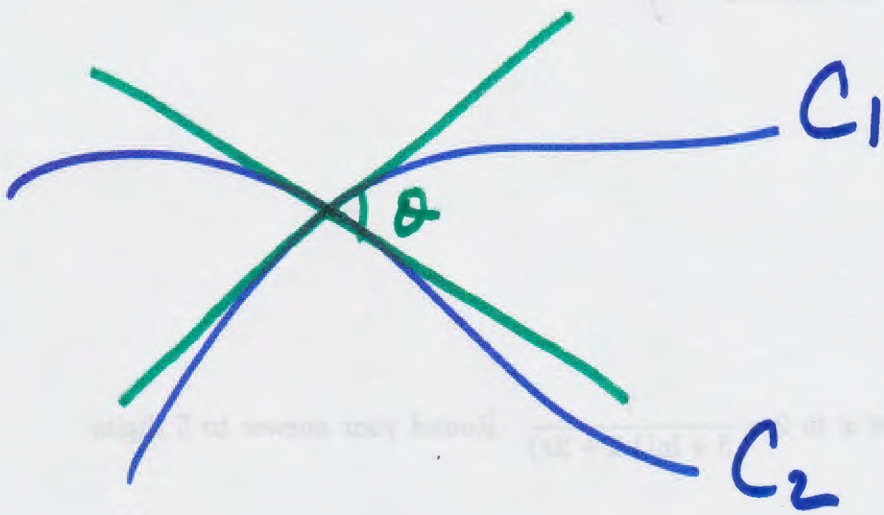
$\cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{29}}\right)$ is an obtuse angle.

Answer:

$$\pi - \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{29}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}\sqrt{29}}\right)$$

Answer:

example: Find the angle between the curves $y=x^2$ and $y=x^3-x$ at their point of intersection.



i.e. we want the angle between their (intersecting) tangents.

First: I need the point of intersection:

$$x^2 = x^3 - x$$
$$0 = x^3 - x^2 - x$$
$$0 = x(x^2 - x - 1)$$

QF: $x = \dfrac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

I'll answer the question for one point only at $x=\frac{1+\sqrt{5}}{2}$

need: **slopes** at $x=\frac{1+\sqrt{5}}{2}$

$y=x^2$

$y'=2x$

$m=2\left(\frac{1+\sqrt{5}}{2}\right)$

$=1+\sqrt{5}$

$y=x^3-x$

$y'=3x^2-1$

$m=3\left(\frac{1+\sqrt{5}}{2}\right)^2-1$

$=\frac{3}{4}(1+2\sqrt{5}+5)-1$

$=\frac{7}{2}+\frac{3}{2}\sqrt{5}$

next: vectors

$\langle 1, 1+\sqrt{5}\rangle$

$\langle 1, \frac{7}{2}+\frac{3}{2}\sqrt{5}\rangle$

OR

$\langle 2, 7+3\sqrt{5}\rangle$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$1 \cdot 2 + (1+\sqrt{5})(7+3\sqrt{5}) = \sqrt{1+(1+\sqrt{5})^2}\,\sqrt{4+(7+3\sqrt{5})^2}\,\cos\theta$$

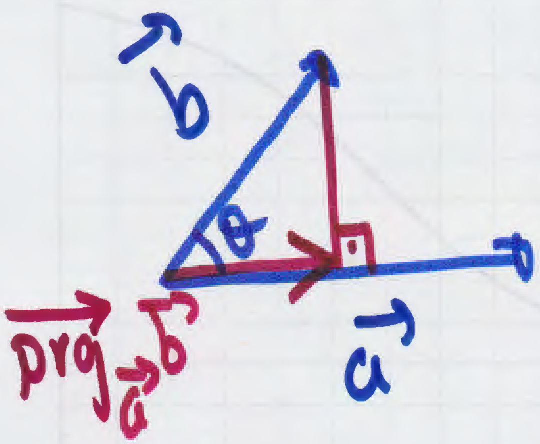
use a calculator

$$\cos\theta = \frac{24+10\sqrt{5}}{\sqrt{1106+490\sqrt{5}}}$$

$$\theta = \cos^{-1}\left(\frac{24+10\sqrt{5}}{\sqrt{1106+490\sqrt{5}}}\right)$$

$$\simeq 0.155 \quad (\simeq 8.87^\circ)$$

Projections (Tuesday's worksheet)



$$\overrightarrow{\text{proj}}_{\vec a}\,\vec b = (\text{length}) \cdot \left(\text{unit vector in the direction of } \vec a\right)$$

check $= \dfrac{\vec a \cdot \vec b}{\vec a \cdot \vec a}\,\vec a$