

Here are the solutions to the curvature exercise.

1. Show that

$$\frac{d}{dt}|\mathbf{v}| = \frac{\mathbf{v} \cdot \mathbf{v}'}{|\mathbf{v}|}.$$

$$\frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})^{1/2} = \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})^{-1/2} \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})^{-1/2}[(\mathbf{v}' \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}')]= \frac{\mathbf{v} \cdot \mathbf{v}'}{|\mathbf{v}|}$$

2. Show that

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{|\mathbf{r}'|^3}[(\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}' - (\mathbf{r}' \cdot \mathbf{r}')\mathbf{r}''].$$

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= \frac{d}{dt} \left(\frac{1}{|\mathbf{r}'|} \mathbf{r}' \right) = \left(\frac{d}{dt} \frac{1}{|\mathbf{r}'|} \right) \mathbf{r}' + \frac{1}{|\mathbf{r}'|} \frac{d}{dt} \mathbf{r}' = -\frac{1}{|\mathbf{r}'|^2} \left(\frac{d}{dt} |\mathbf{r}'| \right) \mathbf{r}' + \frac{1}{|\mathbf{r}'|} \mathbf{r}'' \\ &= -\frac{1}{|\mathbf{r}'|^2} \left(\frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} \right) \mathbf{r}' + \frac{1}{|\mathbf{r}'|} \mathbf{r}'' = -\frac{(\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}'}{|\mathbf{r}'|^3} + \frac{(\mathbf{r}' \cdot \mathbf{r}')\mathbf{r}''}{|\mathbf{r}'|^3} = -\frac{1}{|\mathbf{r}'|^3}[(\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}' - (\mathbf{r}' \cdot \mathbf{r}')\mathbf{r}'']\end{aligned}$$

3. Show that

$$\left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{|\mathbf{r}'|^3}[(\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}' - (\mathbf{r}' \cdot \mathbf{r}')\mathbf{r}''] = -\frac{1}{|\mathbf{r}'|^3}(\mathbf{r}' \times (\mathbf{r}' \times \mathbf{r}''))$$

so taking norms

$$\left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{|\mathbf{r}'|^3} |\mathbf{r}' \times (\mathbf{r}' \times \mathbf{r}'')| = \frac{1}{|\mathbf{r}'|^3} |\mathbf{r}'| |\mathbf{r}' \times \mathbf{r}''| \sin \theta.$$

Since \mathbf{r}' is normal to $\mathbf{r}' \times \mathbf{r}''$, $\sin \theta = 1$. Therefore,

$$\left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2},$$

and then it will follow that

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{\left| \frac{d\mathbf{T}}{dt} \right|}{\left| \frac{ds}{dt} \right|} = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$