Partial fractions - Case III with irreducible quadratic factors and rationalizing

This worksheet completes the lecture on Partial Fractions. When there is an irreducible quadratic (one that cannot be factored into linear factors), the partial fraction expansion has a linear term. That is, for an irreducible $(cx^2 + ex + f)$ factor in the denominator, we have the partial fraction

$$\frac{Ax + B}{cx^2 + ex + f}.$$

1. We wish to integrate

$$\int \frac{7x + 13}{x^3 + 9x} dx.$$

   Factoring the denominator $x^3 + 9x = (x^2 + 9)x$, we have the linear factor x and the irreducible quadratic factor $x^2 + 9$. Note that you cannot factor $x^2 + 9$ any further.

   (a) Write

$$\frac{7x + 13}{x^3 + 9x} = \frac{Ax + B}{x^2 + 9} + \frac{C}{x},$$

   add the fractions on the right, set the numerators equal to each other and solve for the numbers A, B, and C by plugging in values for x.

   (b) Now integrate.

2. What would the partial fraction expression for

$$f(x) = \frac{x^5 + 4x^3 - 7}{x^3(x^2+4)(x+4)^2(x^2+13)(x-2)}$$

be? Do not compute the numbers.

3. When the irreducible quadratic has a linear bx term, we need to eventually complete the square to integrate. The partial fraction algebra is simpler if we complete the square BEFORE doing partial fractions. For example, instead of

$$\frac{x^2+7x+3}{(x^2+2x+5)(x-3)} = \frac{Ex+F}{x^2+2x+5} + \frac{C}{x-3}.$$

do

$$\frac{x^2+7x+3}{(x^2+2x+5)(x-3)} = \frac{x^2+7x+3}{\left[(x+1)^2+4\right](x-3)} = \frac{A(x+1)+B}{(x+1)^2+4} + \frac{C}{x-3}.$$

When you add the fractions on the right, setting the numerators equal you would get

$$x^2+7x+3 = \left[A(x+1)+B\right](x-3) + C\left[(x+1)^2+4\right]$$

Use $x = 3$, $x = -1$ and one more x value to find the numbers A, B, and C.

4. Integrate

$$\int \frac{2x^2 + x + 1}{(x^2 + 6x + 10)(x - 1)} dx.$$

by first completing the square of $x^2 + 6x + 10$ and then doing partial fractions as above.

5. When we have a function with a radical, we can use u- substitution to rationalize. When the integrand contains the expression $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ might work. For the integral

$$\int \frac{1}{(1+x)\sqrt{x}} dx$$

(a) Do the substitution $u = \sqrt{x}$ (or $x = u^2$).

(b) Now integrate the rational function you get and write your answer in terms of x.