Section 3.4 - The Chain Rule

The Chain Rule is for differentiating **compositions of functions**. For example, if

$$h(x) = \sin(x^2)$$

then

$$h(x) = f(g(x))$$

where

$$f(x) = \qquad\qquad \text{and} \qquad\qquad g(x) = .$$

So, can we find

$$\frac{d}{dx} f(g(x))$$

knowing $f'(x)$ and $g'(x)$?

In order to come up with the formula, we will change our notation. Let

$$y = f(g(x))$$

and

$$u = g(x)$$

then

$$y = f(u).$$

We want to compute

$$\frac{dy}{dx}$$

We remember that this was a rate of change (or a slope):

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \frac{\Delta u}{\Delta u} \\
&= \\
&= \\
&= \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}$$

The First Version of The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This is easy to remember because it works like fraction arithmetic. You can apply it to any variables. We use this version mainly in story problems where x, y and u are quantities represented possibly by different letters.

Now, substitute $u = g(x)$, $y = f(u) = f(g(x))$ and switch to prime notation

to get the

The Second Version of The Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

This is the version we use to differentiate formulas.

Example 1:

Use the Chain Rule to compute $\dfrac{d}{dx} \sin\left(x^2\right)$ by first identifying the two functions making up the composition.

Example 2:

Use the Chain Rule to compute $\dfrac{d}{dx} \left(\sin x - x^3\right)^8$

Example 3:

Use the Chain Rule to differentiate $f(x) = e^{x^2}$

Example 4:

Differentiate $f(x) = \sqrt{1 + 5x^2}$

Example 5:

Differentiate $f(x) = e^{\sin(3x^2+6)}$

Example 5:

Differentiate $f(x) = \sqrt{1 + \tan\left(e^x + 5x\right)}$

Example 6:

Differentiate $f(x) = \sin\left(\sqrt{\cos(e^x)}\right)$

Example 7:

Differentiate $f(x) = \left(\dfrac{x^3 + e^{5x}}{\sqrt{x \sin x + \ln x}}\right)^{13}$

The Second and Higher Derivatives

We start with $f(x)$, differentiate to get $f'(x)$ If we differentiate $f'(x)$, we get $f''(x)$.

The second derivative f'', being the derivative of f', tells us how f' (or slopes of f) change and gives us information about the **shape** of the graph of f.

Example 8:

Compute $f''(x)$ if $f(x) = x^3 + e^{7x}$.

Notation:

$$f, f', f'', f''', f^{iv}, f^{v},$$

After a while Roman numerals gets tricky so we switch to regular numbers. In order to differentiate (in the English sense as in tell apart 🙂) powers of f from derivatives of f , we use parentheses:

$$f, f', f^{(2)}, f^{(3)}, f^{(4)}, f^{(5)}, ...$$

We use third or higher derivatives in Math 126. They are here mainly for notation and computation.

We also have the quotient notation. The second derivative is the derivative of the first derivative:

$$\frac{d}{dx}\frac{dy}{dx} =$$

so we have

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ...$$

Example 9:

Compute $f^{(2021)}(x)$ if $f(x) = x^3 + e^{7x} + \cos x$.

The Derivative of $f(x) = a^x$

We tackled this question in Section 3.1, but got stuck at the limit

$$\lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

where e was the special number with $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ so $\frac{d}{dx}e^x = e^x$.

Now, we will differentiate $f(x) = a^x$ using this, the chain rule and a property of exponentials and logarithms. So, before we start, here is a review of laws of exponentials and logarithms. Most must be familiar, some we will use again in the future.

LAWS OF EXPONENTIALS AND LOGARITHMS

1. $e^x = y$ is the same as saying $\ln y = x$. In other words $f(x) = e^x$ and $g(x) = \ln x$ are **inverse functions**. Therefore,
$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln x} = x.$$
In particular, $\ln e = 1$.
2. $\ln(AB) = \ln A + \ln B$.
This follows from the law of exponentials : $e^x \cdot e^y = e^{x+y}$.
3. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$.
This follows from the law of exponentials : $\frac{e^x}{e^y} = e^{x-y}$.
4. $\ln\left(A^B\right) = B \cdot \ln A$.
This follows from the law of exponentials : $(e^x)^y = e^{xy}$.
5. $\log_A B = \frac{\ln B}{\ln A}$.

We will use 2,3 and 4 in **logarithmic differentiation** later.

Now, we are ready to compute the derivative of a^x by rewriting it in terms of e:

Therefore,
$$\frac{d}{dx}a^x = \ln a \cdot a^x$$
which also tells us

$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a$.

Example 10:

$\frac{d}{dx}3^{\tan x}$