Section 3.3 - Derivatives of Trigonometric Functions

The rules are simple and you can keep them on your note sheet until you memorize most. Once you know

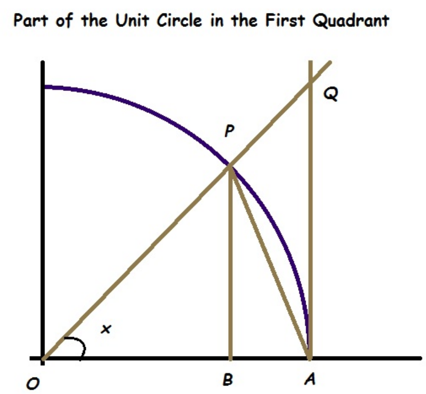
- $$\frac{d}{dx}\sin x = \cos x$$
- $$\frac{d}{dx}\cos x = -\sin x$$

you can get the other four derivatives using the quotient rule. However, there are lessons to be learned from why they are the way they are, so we will derive them before we look at examples.

An Important Limit

We will use the definition of the derivative to get the derivative of $\sin x$. However, at some point, we will have two interesting limits to evaluate. One has a geometric proof. Once you understand and appreciate this, you can forget the details - but not the resulting limit.

Below is part of the unit circle. We start by recalling the definitions of the trigonometric functions $sinx$, $\cos x$ and $\tan x$ and placing them on the picture.



Then looking at the picture, we make a comparison of areas:

$$\text{area of } \Delta OAP \leq \text{area of slice of disc } OPA \leq \text{area of } \Delta OAQ$$

Now, replace them by their values in terms of x and the three trigonometric functions $\sin x$, $\cos x$ and $\tan x$:

After reorganizing, we should get

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Now, we use the Sandwich (Squeeze) Theorem to take the limits:

$$\lim_{x\to 0} \cos x \leq \lim_{x\to 0} \frac{\sin x}{x} \leq \lim_{x\to 0} 1.$$

We have this important limit:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example 1:

Use the limit above to compute $\lim_{h\to 0} \dfrac{\cos h - 1}{h}$.

Example 2:

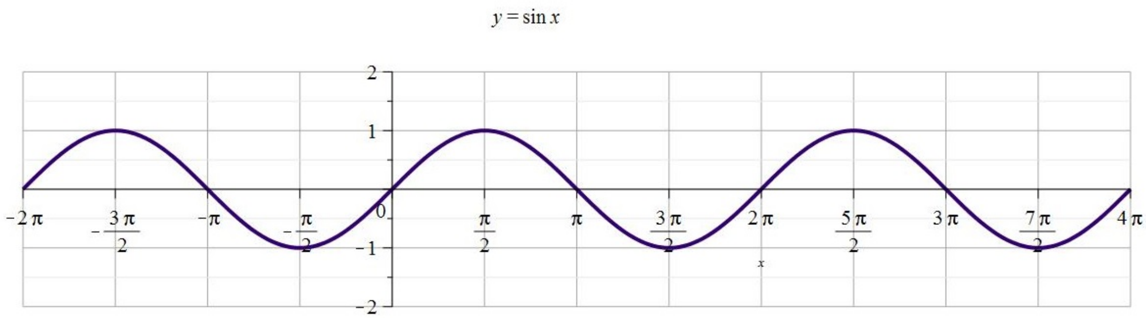
Compute $\lim_{x\to 0} \dfrac{\sin(3x)}{4x}$.

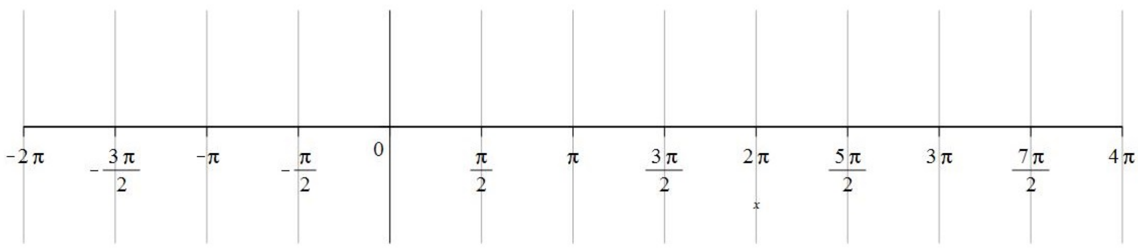
Example 3:

Compute $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin(7x)}$.

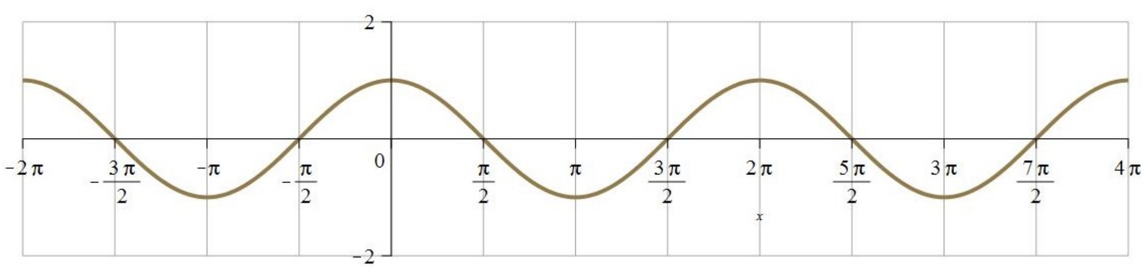
Derivatives of Sine and Cosine

Now, we can tackle the derivative of $\sin x$. We'll do this graphically, make our best guess, and then verify our guess with another limit computation. Here is the graph of $y = \sin x$





Here is a computer generated one



So, if the slope we marked as m is 1, our best guess is that the derivative is $\cos x$. But, the slope m is the limit we worked with earlier, so it is 1. Therefore,

$$\frac{d}{dx} \sin x = \cos x$$

We can also compute the derivative of $\sin x$ using the definition of the derivative. It is in the book if you want to read it. Here is the similar computation for the derivative of $\cos x$. It uses the addition formula for the cosine function

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

together with the two limits we have seen above. You do not have to remember this derivation.

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{x\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0}\left(\frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}\right) \\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0}\frac{\sin x \sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0}\frac{(\cos h - 1)}{h} - \sin x \cdot \lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 = -\sin x
\end{aligned}$$

Therefore, we have

$$\frac{d}{dx}\cos x = -\sin x$$

With these two, and the quotient rule, we can get the derivative of $\tan x$:

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

Exercise:

Compute the following:

1. $\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\tan x}$

2. $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$

3. $\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x}$

Now, check your work with the formulas in the book (or the internet). Finally, we can combine our rules from the first two sections to differentiated more complicated formulas:

Example 4:

Differentiate $f(x) = \dfrac{e^x \cos x}{x^2 + 4}$

Example 5:

Differentiate $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$