Section 3.2 - Product and Quotient Rules

We will first start with stating the two rules, then do several examples. At the end, if you want, you can see why they are true. You are not responsible for knowing why they are true, but for using the rules correctly when you have to.

The Product Rule

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

The Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

Example 1:

Differentiate $f(x) = xe^x$.

Example 2:

Differentiate $f(x) = (x^3 + 4x^2 - 7)(5x^2 + 1)$.

Example 3:

Differentiate $f(x) = e^x(9\sqrt{x} + 1)(8x - 11)$.

Example 4:

Differentiate $f(x) = \dfrac{8x+3}{x^2+5}$.

Example 5:

Differentiate $f(x) = \dfrac{x^3 e^x}{x^2+9}$.

Example 6:

Differentiate $f(x) = \dfrac{x^2+5x}{\sqrt{x}}$.

Example 7:

Find the equation of the tangent line to the graph of $y = \dfrac{x}{1+x^3}$ at the point where $x = 1$.

This part is optional. You are not responsible for its content.

Why is the product rule the way it is?

In order to differentiate $f(x) \cdot g(x)$ we can use the Definition of the Derivative:

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Now we do a "trick", subtracting and adding the same term

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

and then we regroup terms

$$= \lim_{h \to 0} \frac{(f(x+h) - f(x)) \cdot g(x+h) + f(x) \cdot (g(x+h) - g(x))}{h}$$

we do algebra to separate into two fractions

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right)$$

now we use the properties of the limits to write this as a combination of four limits

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

and, finally, we evaluate the limits

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Why is the quotient rule the way it is?

Doing this with the definition of the derivative like above is possible, but, much messier due to fractions. Here is a simpler way to do it:

$$f' = \left(\left(\frac{f}{g} \right) g \right)' = \left(\frac{f}{g} \right)' g + \left(\frac{f}{g} \right) g'$$

where we get the second equality from the product rule. Now solve for the $\left(\frac{f}{g} \right)'$ in

$$f' = \left(\frac{f}{g} \right)' g + \left(\frac{f}{g} \right) g'$$

and simplify to get the quotient rule.