

Section 3.1 - Differentiation Rules

We defined the derivative of $f(x)$ with the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Using this definition is not easy and it gets more difficult when the formula for $f(x)$ is complicated. Can you imagine how long it would take you (let alone whether it is even possible) to use this limit definition to compute the derivative of

$$f(x) = \left(\frac{x^3 + e^{5x}}{\sqrt{x \sin x + \ln x}} \right)^{13} ?$$

So, what we need are rules of two types:

- Rules that tell us what derivatives of basic formulas like x^a , e^x , $\sin x$, $\cos x$, $\ln x$, $\sin^{-1} x$, etc. are.
- Rules that tell us what to do with sums, differences, products, quotients and compositions of functions.

In this section we start with the following:

1. $\frac{d}{dx} x^a =$
2. $\frac{d}{dx} e^x =$
3. $\frac{d}{dx} (f(x) + g(x)) =$
4. $\frac{d}{dx} (c \cdot f(x)) =$

Derivatives of Exponential Functions

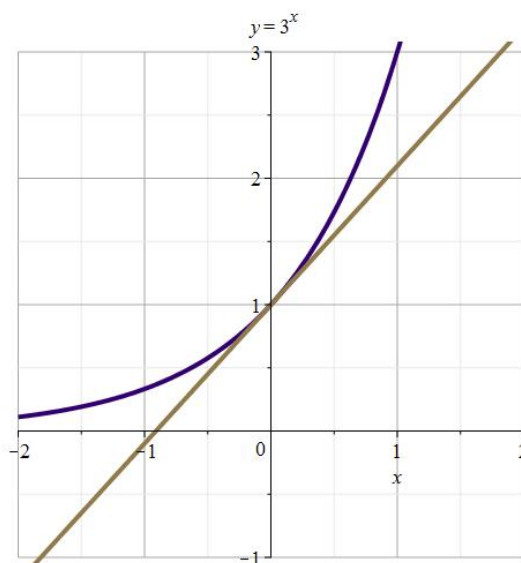
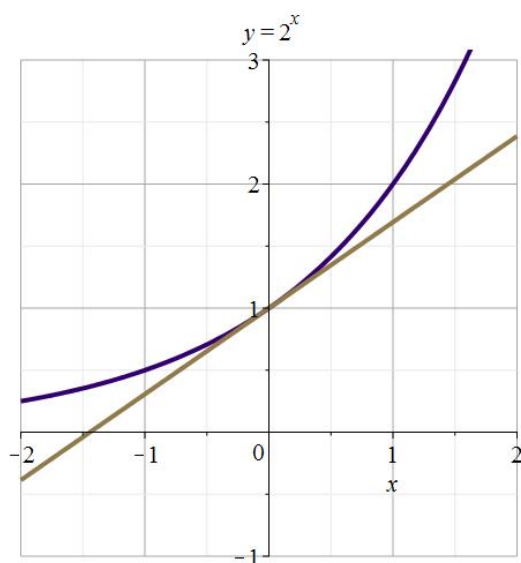
We want to compute the derivative of $f(x) = a^x$. We start with the definition of the derivative:

$$\lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} =$$

So we will be able to get a formula for the derivative of $f(x) = a^x$ IF we know the slope of the tangent line at $x = 0$ which is given by the limit

$$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

For example, we have the graphs of $f(x) = 2^x$ and $f(x) = 3^x$ below.



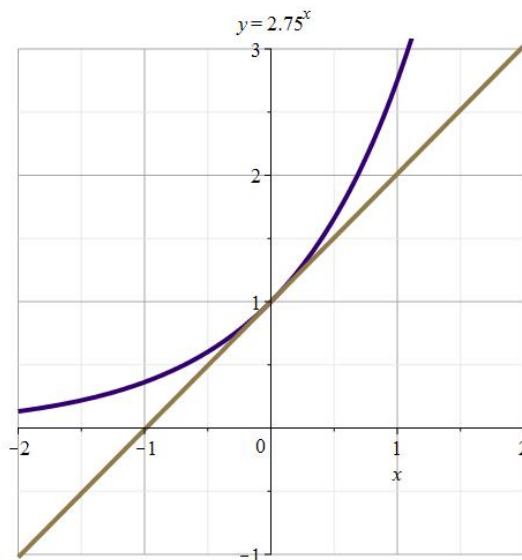
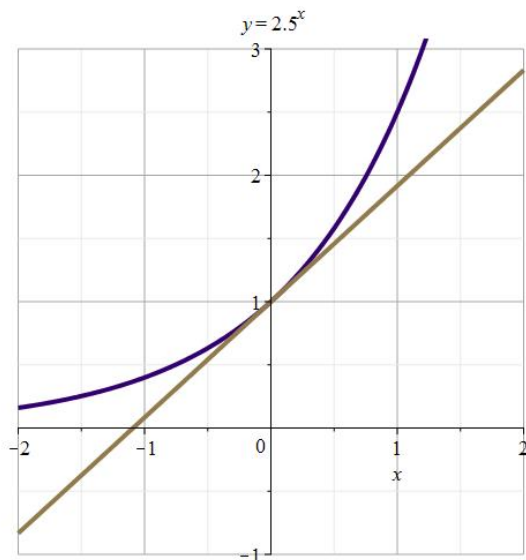
Use a calculator to approximate the limits

$$\lim_{h \rightarrow 0} \frac{2^h - 1}{h} \approx$$

$$\lim_{h \rightarrow 0} \frac{3^h - 1}{h} \approx$$

You can see one is less than 1 and the other is greater than 1. The limits are the slopes of the tangent lines you see above which you can also approximate using the graphs.

Below, we have the graphs of $f(x) = 2.5^x$ and $f(x) = 2.75^x$



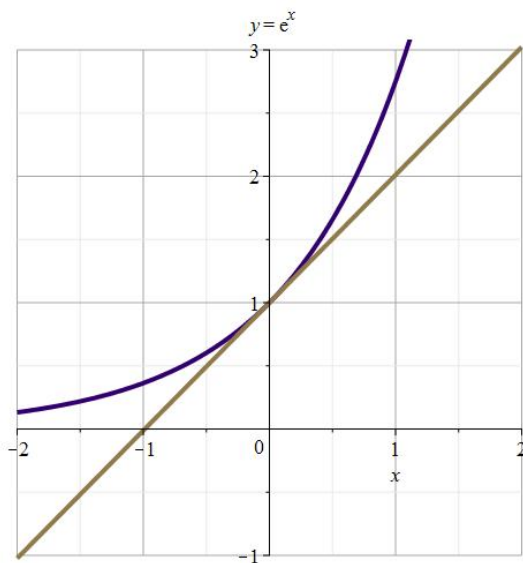
Use a calculator to approximate the limits

$$\lim_{h \rightarrow 0} \frac{2.5^h - 1}{h} \approx$$

$$\lim_{h \rightarrow 0} \frac{2.75^h - 1}{h} \approx$$

You can see one is less than 1 and the other is greater than 1. The limits are the slopes of the tangent lines you see above which you can also approximate using the graphs. In fact, the second one is ALMOST 1, which you can tell either looking at the graph or the limit approximation with the calculator.

So, there must be a number, which is called e , between 2.5 and 2.75, such that the graph of e^x has slope EXACTLY 1 at the point where $x = 0$!



So,

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

and going back to the first argument above

$$\lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

We have our first rule

$$\frac{d}{dx} e^x = e^x.$$

The question of the derivative of $f(x) = a^x$ for an arbitrary $a > 0$ cannot be answered at this point. We will come back to it.

Derivatives of Powers Functions

Now, we tackle the $f(x) = x^a$:

When $a = 0$,

$$\frac{d}{dx} x^0 = \frac{d}{dx} 1 =$$

When $a = 1$,

$$\frac{d}{dx} x^1 = \frac{d}{dx} x =$$

When $a = 2$,

$$\frac{d}{dx} x^2 = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} =$$

When $a = 3$,

$$\frac{d}{dx} x^3 = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} =$$

For the general case, you need to know how $(x+h)^n$ foils out. Luckily, we don't need the complete expansion. Here is what will happen with $n = 5$. We get the coefficients from *Pascal's Triangle*

$$\begin{array}{ccccccc} & & & & 1 & & & \\ & & & & 1 & & 1 & \\ & & & 1 & & 2 & & 1 \\ & & 1 & & 3 & & 3 & & 1 \\ & 1 & & 4 & & 6 & & 4 & & 1 \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

so,

$$\frac{d}{dx} x^5 = \lim_{h \rightarrow 0} \frac{(x+h)^5 - x^5}{h} =$$

So for any positive integer n , in fact, for any number n , we have

$$\frac{d}{dx} x^n = nx^{n-1}$$

The last two rules we will state without explanation:

- The derivative of a sum of two functions is the sum of their derivatives:

$$\frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x)$$

- The derivative of a constant multiple of a function is the constant multiple of its derivative:

$$\frac{d}{dx} (c \cdot f(x)) = c \cdot f'(x)$$

Now we are ready for examples.

Example 1:

Differentiate $f(x) = 6x^3 - 7x^2 + 12x - 99$.

Example 2:

Differentiate $f(x) = \sqrt{x}$.

Example 3:

Differentiate $f(x) = \frac{1}{x}$.

Example 4:

Differentiate $f(x) = x^{1/3}$.

Example 5:

Differentiate $f(x) = e^x + \sqrt{6x} - \frac{3}{x} + \frac{1}{2\sqrt{x}}$.

Example 6:

Differentiate $f(x) = \frac{2x + 9}{\sqrt[3]{x}}$.

Example 7:

Find the equation of the tangent line to the graph of $f(x) = 3x^2 - 7\sqrt{x}$ at the point where $x = 4$.