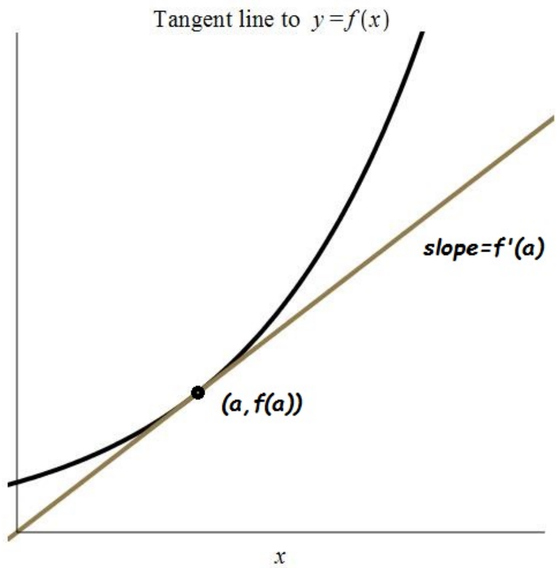
Section 2.8 - The Derivative as a Function

We defined the derivative at $x = a$ to be the slope of the tangent line to the graph of $y = f(x)$ at the point where $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Today, we would like to vary the point of tangency and see how the derivative which gives the slope of the tangent line varies. So, to have this new point of view, we will use

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

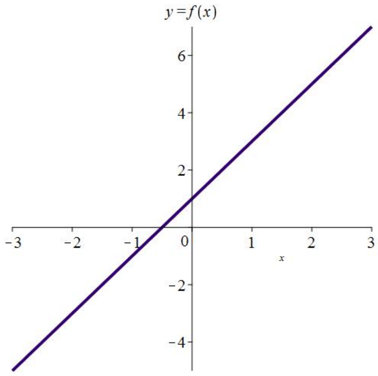
This is now the **derivative function** $f'(x)$ and its value changes with x.

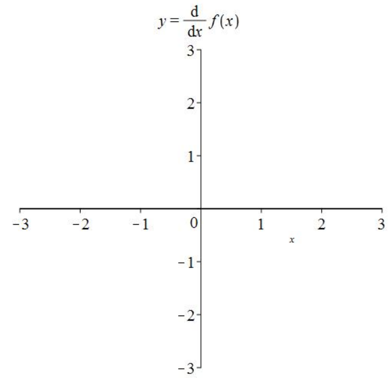
Example 1:

If we have a **linear** function like

$$f(x) = 2x + 1$$

its graph is a line and it is its own tangent line at any point.





You can also compute the derivative using the definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

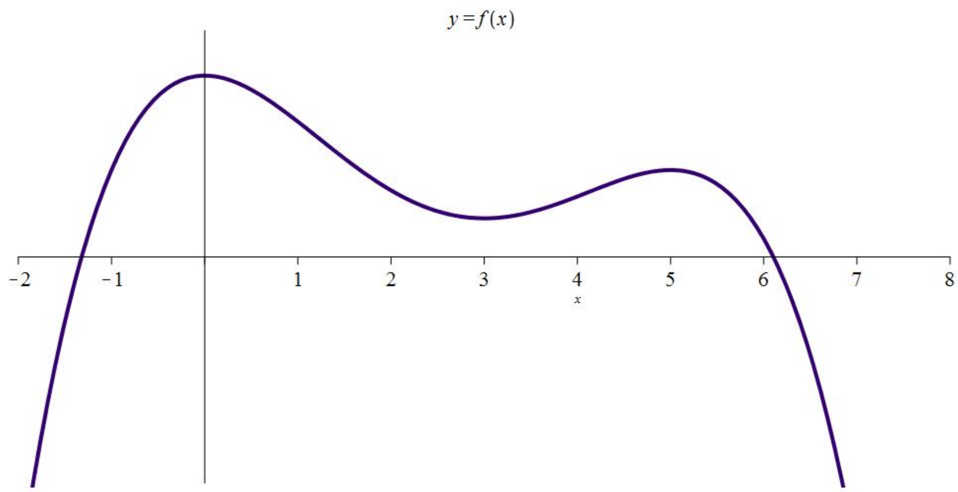
In general, for any linear function $f'(x) = mx + b$, the derivative function is constant and equals the slope of the line

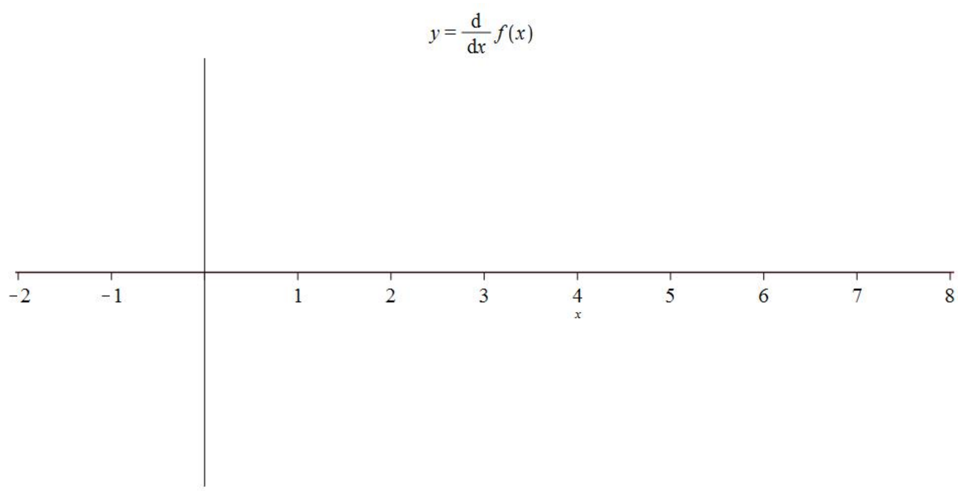
$$f'(x) = m.$$

What makes a line a line is that the slope is constant.

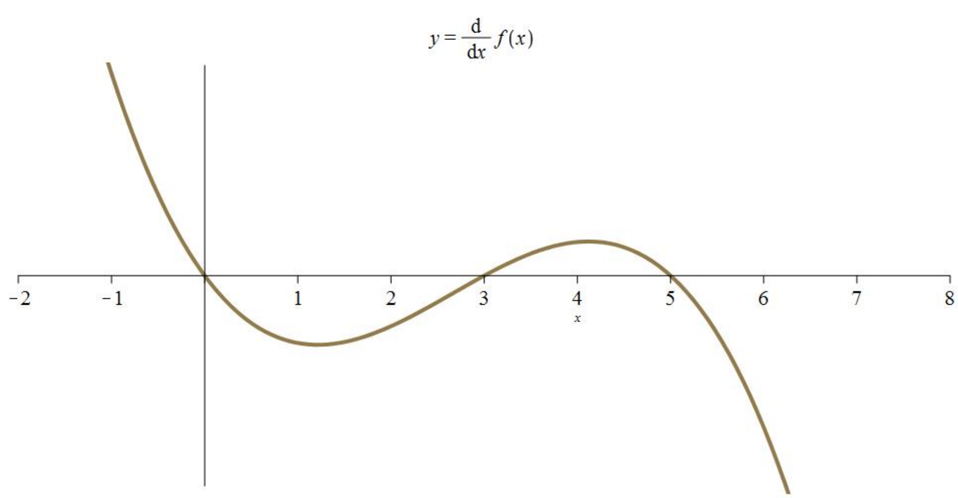
Example 2 - Graphical

Use the graph of $y = f(x)$ below to sketch a graph of $y = f'(x)$. Note that y-axes are not scaled so we are not interested in actual values, but the **shape** of the derivative:





Here is the computer generated version. Does your answer look similar?



To sketch the derivative function $y = f'(x)$ from a graph of the function $y = f(x)$ we look at the following details:

1. If the graph of the function has a **horizontal tangent**, then the derivative $f'(x) = 0$ at those x values. This is because horizontal lines have slope zero. These are good points to get started on your derivative graph.

2. If the graph of $y = f(x)$ is **increasing**, tangent lines will have **positive slopes**, so the values of the derivative $f'(x)$ will be **positive**. You graph the derivative above the x-axis.

   If the graph of $y = f(x)$ is **decreasing**, tangent lines will have **negative slopes**, so the values of the derivative $f'(x)$ will be **negative**. You graph the derivative below the x-axis.

3. You can also see if the values of $f'(x)$ are increasing

   or decreasing

   by looking at the shape of the graph of $y = f(x)$.

Notation

We can also think of $f'(x)$ as rate of change of $f(x)$. For example, if $h(t)$ is a height function of an object, $h'(t)$ will be the velocity of the object. If $V(t)$ is the volume, $V'(t)$ is the rate of change of volume. since the derivative is defined as a slope (quotient), the derivative will have units define by

$$\frac{\text{units of } f}{\text{units of } x}$$

or

$$\frac{\text{units of } h}{\text{units of } t}$$

etc.
We also have a notation for the derivative which reminds that is was a quotient in the first place:

$$\text{slope of secant} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

so

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

If $y = f(x)$, the following are all the same:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$$

What does it mean for a function to be differentiable?

Remember the derivative is defined as a **limit**, so we say a function is differentiable at x if the limit

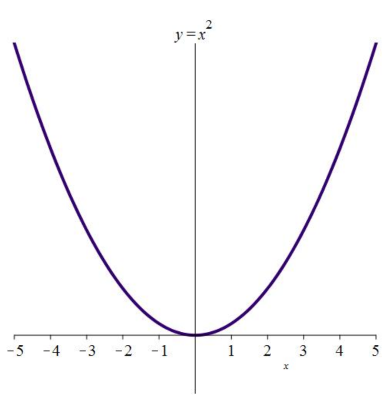
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. So far we have not seen any examples where it did not. What can go wrong?

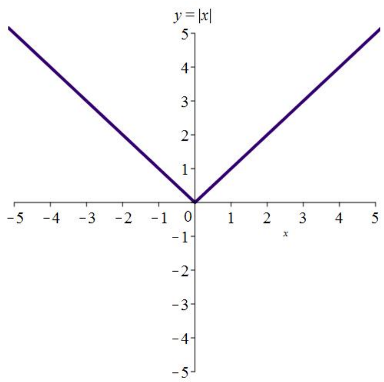
Geometrically, a function is differentiable at a point if it has a tangent line with a well defined slope. This means two things:

- When you zoom in to the graph, you see a line, which is the tangent line.
- The line has a slope so it is **not vertical**.

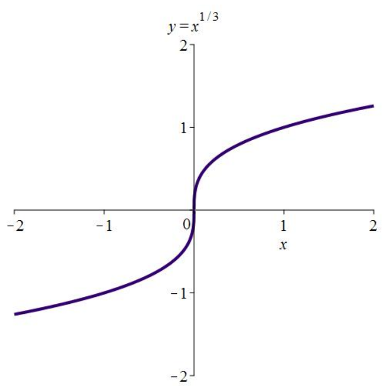
Example 3



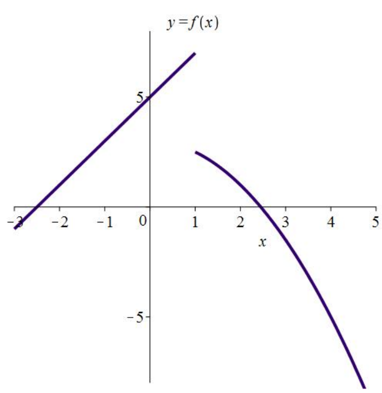
Example 4



Example 5



Example 6



Theorem:

If f is differentiable at a, then f is continuous at a.