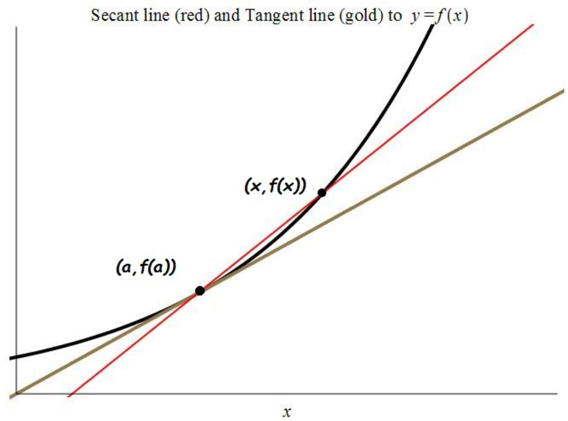
Section 2.7 - Derivatives and Rates of Change

In Section 2.1, we computed the **slope of the tangent line** to the graph of $y = \dfrac{2}{x}$ at the point $(1, 2)$ by looking at slopes of secant lines. Today, we come back to this problem and repeat the process for an arbitrary function and a point of tangency.

To compute any slope, we need two points on the line, however, in the case of the tangent line, we only had the point of tangency, say $(a, f(a))$. Our solution was to take a second point on the graph as $(x, f(x))$ and to compute the slope of the line through $(a, f(a))$ and $(x, f(x))$ as

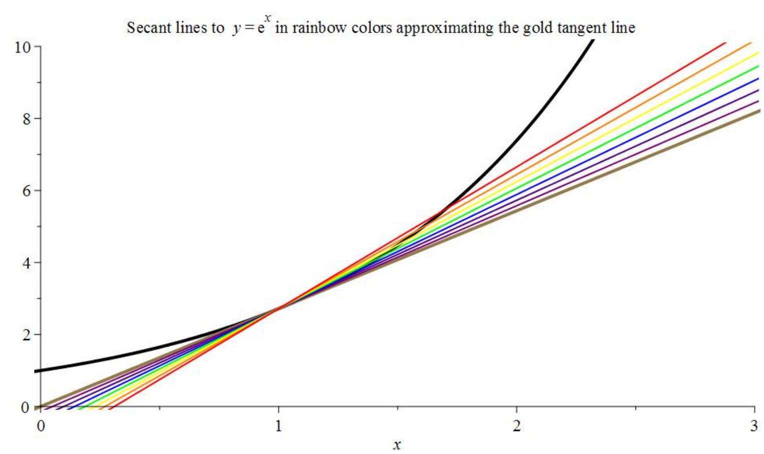
$$\text{slope} = \frac{f(x) - f(a)}{x - a}.$$



Then, we looked to see what happened to this quotient when values of x were getting closer and closer to a. Now that we have the concept of a limit, what we did (using a table back then) was that we computed

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Below is a graph of $y = f(x)$, with several of its tangents showing how they get close to the (gold) tangent line as their second point approaches the point of tangency.



The BID IDEA is that the slope of the secant lines would get close to the **slope of the tangent line** as $x \to a$. It is so big, that it gets a new name and notation:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is called the **derivative of $f(x)$ at $x = a$**

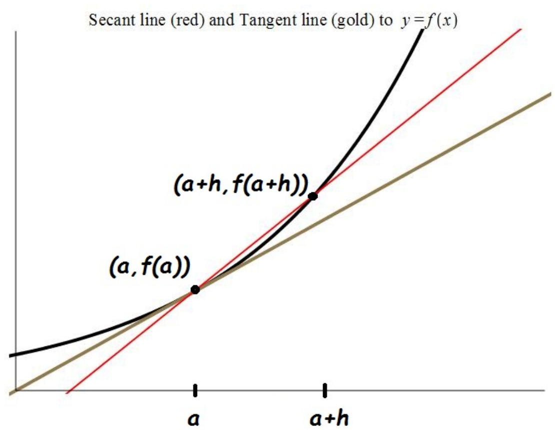
Example 1:

Find the slope and the equation of the tangent line to the parabola

$$y = 3x - 7x^2$$

at the point where $x = 2$ using the limit definition above.

Here is an alternative notation (which makes the algebra simpler sometimes)



$$\text{Slope of the Secant Line} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

$$\text{Slope of the Tangent Line} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 2:

Compute $f'(a)$ if $f(x) = 7x^2 + 5x + 3$.

The Derivative as Rate of Change

If $f(t)$ is distance, for example, and we want to compute the (instantaneous) velocity at $t = a$, we can proceed like above.

First, we compute the average velocity from $t = a$ to $t = a + h$ which is

$$\frac{\textbf{change in distance}}{\textbf{change in time}} = \frac{f(a+h) - f(a)}{(a+h) - h} = \frac{f(a+h) - f(a)}{h}$$

and then we compute the (instantaneous) velocity at time a which is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

If f is in meters and t is in seconds, looking at the quotient, the velocity $f'(a)$ will be in meters per second, or m/s.

This applies to any quantity, not necessarily distance.

Given $f(x)$, its derivative $f'(x)$ gives the rate of change of f.

Example 3:

The volume of something is given by

$$V(t) = \frac{1}{1+t}$$

in cubic meters where t is in minutes. Compute the instantaneous rate of change of volume at time $t = 2$. Include units with your answer.

Example 4:

Use the limit definition of the derivative to compute $f'(1)$ if

$$f(x) = \sqrt{x^2 + 1}.$$

Use it to write the equation of the tangent line to the graph of $y = \sqrt{x^2 + 1}$ at the point where $x = 1$.