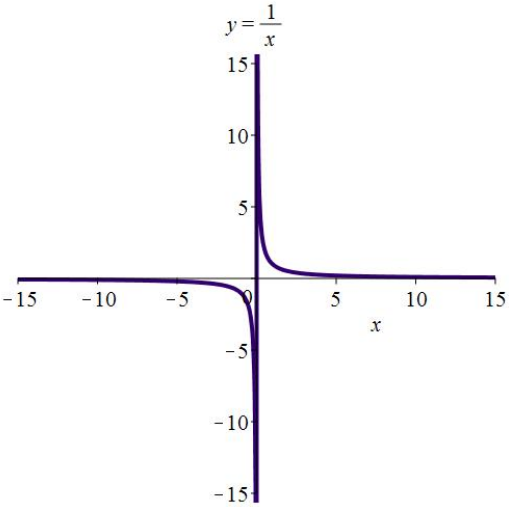
Section 2.6 - Limits at Infinity, Horizontal Asymptotes

Example 1:

What happens to $f(x) = \frac{1}{x}$ when the values of x get VERY large?



We say

$$\lim_{x\to\infty} f(x) = L$$

if the values $f(x)$ get arbitrarily close to the number L when x is large enough.

Similarly,

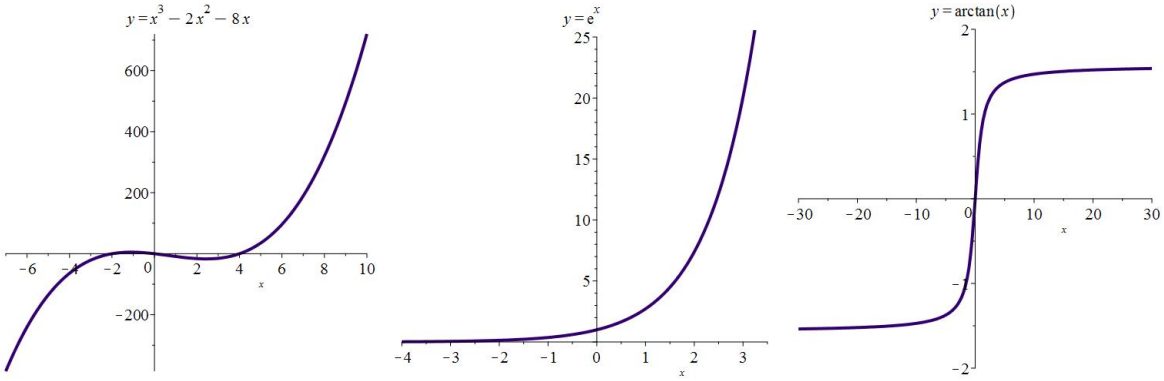
$$\lim_{x\to-\infty} f(x) = L$$

if the values $f(x)$ get arbitrarily close to the number L when x is negative and large enough.

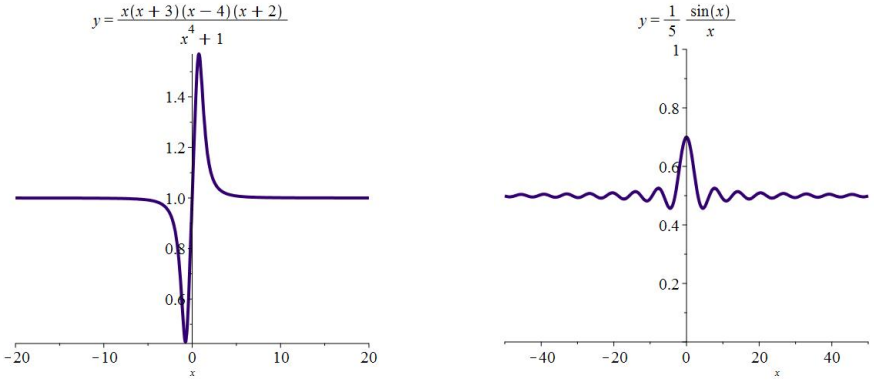
If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a **horizontal asymptote**. The graph of $y = f(x)$ will approach the line $y = L$ on the side(s) with the limit L.

Questions:

1. How many horizontal asymptotes can a function have?
   (a) Can you find an example of a function (graph or formula) which has no horizontal asymptotes?
   (b) Can you find an example of a function (graph or formula) which has one horizontal asymptote?
   (c) Can you find an example of a function (graph or formula) which has two horizontal asymptotes?
   (d) Can you find an example of a function (graph or formula) which has three or more horizontal asymptotes?
2. Can the graph of a function cross its horizontal asymptote?



Graphs of functions crossing their horizontal asymptotes:



You can see the horizontal asymptotes from the pictures. But, if were only given the formulas, how would be do the limit computations?

Limit computations as $x \to \infty$ and $x \to -\infty$

Example 2:

$$\lim_{x\to\infty} 3x + 7$$

and

$$\lim_{x\to-\infty} 3x + 7$$

Example 3:

$$\lim_{x\to\infty} -4x^2 + 1000x + 250$$

and

$$\lim_{x\to-\infty} -4x^2 + 1000x + 250$$

General Rule: When x gets very large (as it will when $x \to \infty$ or $x \to -\infty$, the highest power dominates.

So when we look at limits of polynomials as $x \to \infty$ or $x \to -\infty$, we only look at the leading term with the highest power.

Example 4:

$$\lim_{x \to -\infty} 3x^5 + 10000x^4 + 40000000x^2 + 700000000000$$

So we can say that graphs of polynomials will not have horizontal aysmptotes (except for constant functions, polynomials of degree 0, which are their own asymptotes)

Example 5:

$$\lim_{x \to \infty} e^x$$

and

$$\lim_{x \to -\infty} e^x$$

General Rule for $y = a^x$:

Well, it depends on whether $0 < a < 1$ or $a > 1$:

Example 6:

$$\lim_{x \to \infty} \frac{1}{x^3}$$

and

$$\lim_{x \to -\infty} \frac{1}{x^3}$$

General Rule for $y = \frac{1}{x^r}$ with $r > 0$:

Example 7:

$$\lim_{x\to\infty} \frac{2x^3 + 5x - 7}{5x^3 + 4x^2 + 6}$$

Example 8:

$$\lim_{x\to\infty} \frac{x^2 + 4}{5x + 7}$$

Example 9:

$$\lim_{x\to\infty} \frac{5x^2 + 4}{3x^3 + 2x + 1}$$

General Rule for rational functions (quotients of two polynomials):

Divide the numerator and the denominator by the highest power you see and re-evaluate the limit.

Think about what would happen if the numerator has higher degree than the denominator, if the denominator has higher degree than the numerator, if they have the same degree.

Example 10:

$$\lim_{x\to\infty} \frac{\sqrt{9x^2 + 4}}{x + 4}$$

and

$$\lim_{x\to-\infty} \frac{\sqrt{9x^2 + 4}}{x + 4}$$

Example 11:

$$\lim_{x\to\infty} \sqrt{9x^2+7x} - 3x$$

and

$$\lim_{x\to-\infty} \sqrt{9x^2+7x} - 3x$$