Section 2.5 - Continuity

We have mentioned several times how some functions have their **limits equal to their function values**. Such functions are called **continuous**. More precisely,

We say that $f(x)$ is continuous at $x = a$ if

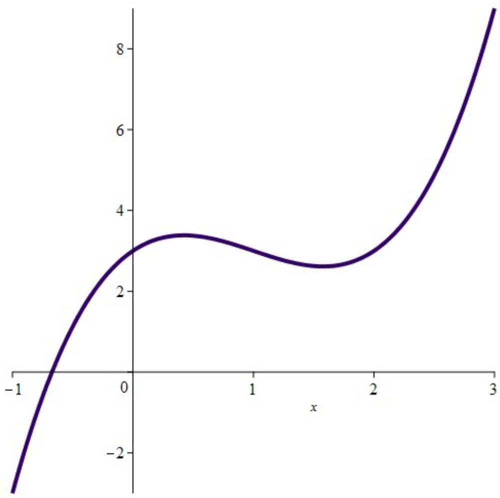
$$\lim_{x \to a} f(x) = f(a).$$

Terminology: Removable discontinuity, jump discontinuity, infinite discontinuity, left continuous, right continuous.

Examples with Graphs:

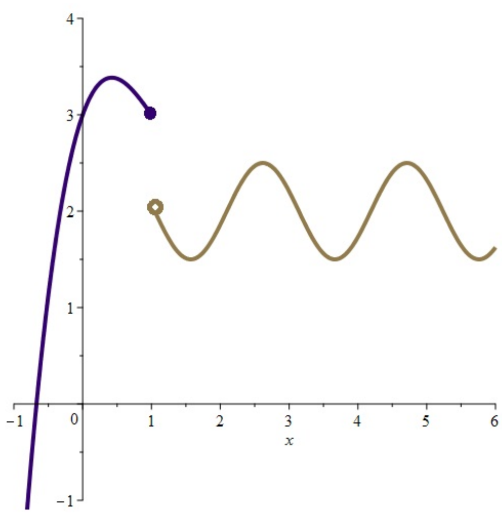
Example 1:

Is $f(x)$ continuous at $x = 1$? Why or why not?



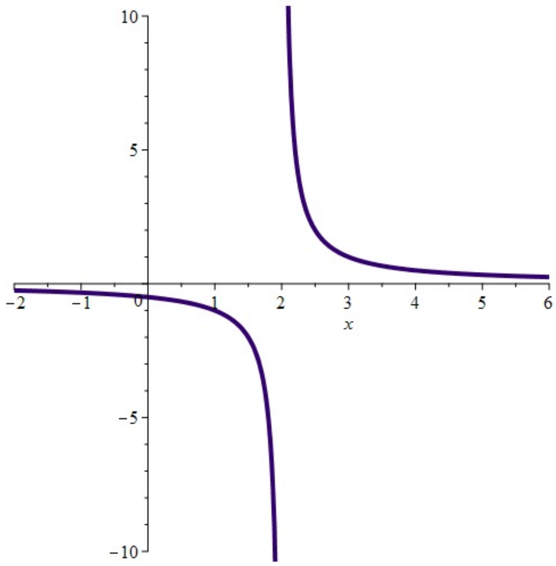
Example 2:

Is $f(x)$ continuous at $x = 1$? Why or why not?



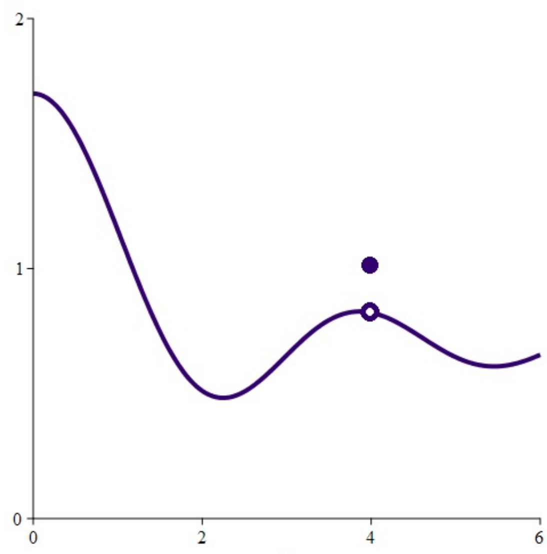
Example 3:

Is $f(x)$ continuous at $x = 2$? Why or why not?



Example 4:

Is $f(x)$ continuous at $x = 4$? Why or why not?



Examples with Formulas:

Here we first evaluate the limit. If the limit exists, we compare it with the function value.

Example 5:

Is $f(x) = \dfrac{3x^2 - 2x + 4}{x + 4}$ continuous?

Direct Substitution Property
If $x = a$ is in the **domain** of an algebraic function, then

$$\lim_{x \to a} f(x) = f(a).$$

So now we can reword this as:

Algebraic functions are continuous on their domains.

Example 6:

Is the function defined by

$$f(x) = \begin{cases} x + 4 & \text{if } x \leq 0 \\ x^2 + 2 & \text{if } x > 0 \end{cases}$$

continuous?

Example 7:

Is $f(x) = \dfrac{1}{x - 4}$ continuous?

Example 8:

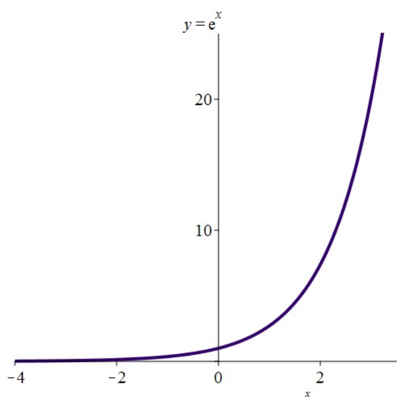
Is the function $f(x) = \dfrac{x^3 - 1}{x - 1}$ continuous? What is its domain?

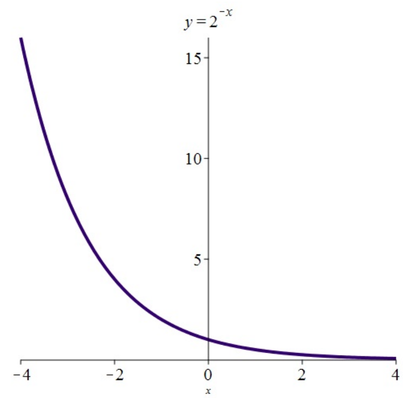
So far we only talked about and used examples of algebraic functions. But, we also have **exponential** and **logarithmic** functions, **trigonometric** and **inverse trigonometric** functions. Here is a brief review of their basic properties, graphs and continuity.

Exponential Functions

$$f(x) = a^x, \;\; a > 0$$

Examples are $f(x) = 2^x$ and $f(x) = e^x$, where the latter is commonly used. Domains are ALL real numbers. They are **continuous everywhere.**

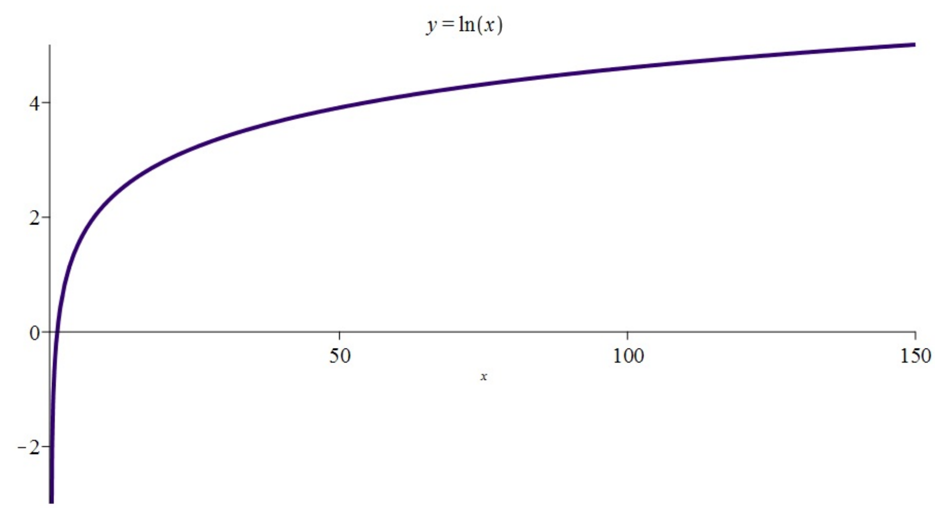




Logarithmic Functions

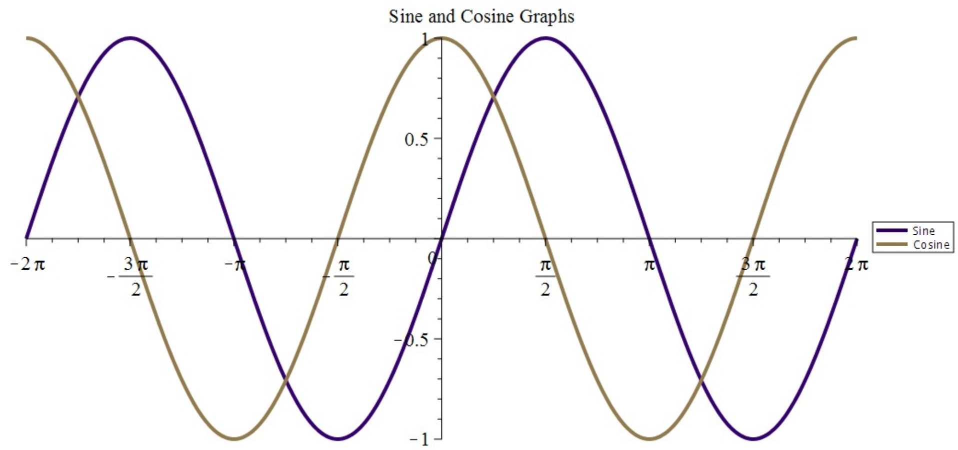
$$f(x) = \log_a x, \;\; a > 0$$

We almost always work with the **natural logarithm** function $f(x) = \ln x$ which is really $f(x) = \log_e x$. Its domain is POSITIVE real numbers. It is continuous when $x > 0$ where it is defined.

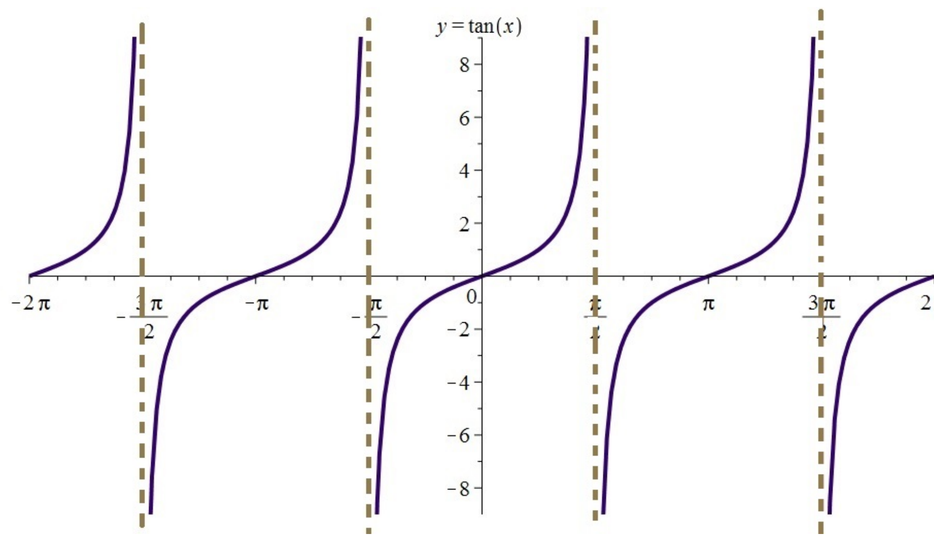


Trigonometric functions

Sine and cosine have domain ALL real numbers and they are **continuous everywhere**.



The other four trigonometric functions are defined as quotients. They are defined and continuous everywhere except when the denominator is 0. At the points where the denominator is 0, there are **infinite discontinuities**. For example, $f(x) = \tan x$ has infinite discontinuities where $\cos x = 0$ giving the the vertical asymptotes on the graph.



Inverse trigonometric functions are also continuous on their domains (where they are defined.

It turns out that any combination of continuous functions are continuous. We can write this in two parts:

1. Sums, differences, products or continuous functions are continuous. The quotient of two continuous functions is continuous where the denominator is not 0. (Thus follows from the Limits Laws which says the limit of a sum/difference/product/quotient is the sum/difference/product/quotient of limits as long as the denominator in a quotient is not 0.
2. Compositions of continuous functions are continuous. (This follows again from a similar theorem in limits.)

In practice, as long as you can evaluate a formula - calculator is not giving you an error so you are not dividing by zero, taking the square root of a negative number, or taking the logarithm of a non-positive number- the limit will be the function value!

Example 7:

The function $f(x) = \dfrac{x\sin(x^2+1) - \sqrt{x+9}}{e^x(x^3-1)}$ is continuous everywhere where $x \geq -9$ except at $x = 1$. For example,

$$\lim_{x\to 0} \frac{x\sin(x^2+1) - \sqrt{x+9}}{e^x(x^3-1)} =$$

Example 8:

What is

$$\lim_{x\to 0} \frac{\sin(x)}{x} =?$$