

Section 2.3 - Calculating Limits Using Limit Laws

In this section we'll see how we can compute limits using algebra precisely, without resorting to calculators. We'll also see which category of functions have their limits equal to their function values. For the others where we cannot plug in the values of a into the formula, we'll see some algebra tricks that will help up evaluate limits.

Example 1:

$$\lim_{x \rightarrow 3} (3x^2 + 5x - 17) =$$

This was almost common sense! Well, that common sense had to be justified mathematically before it could be used. Luckily, we won't be going through that process (that math majors have to learn), but we will simply list the results. Again, these are mostly common sense so you will not have to memorize them. We just take them for granted.

Limit Laws

If

$$\lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a} g(x) = M$$

(so both limits exist), then

1. $\lim_{x \rightarrow a} c = c$

2. $\lim_{x \rightarrow a} x = a$

3. $\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$ and $\lim_{x \rightarrow a} [f(x) - g(x)] = L - M$

4. $\lim_{x \rightarrow a} c \cdot f(x) = c \cdot L$

5. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = LM$

6. By repeatedly using the previous law we get $\lim_{x \rightarrow a} [f(x)]^n = L^n$ for $n = 1, 2, 3, \dots$

7. If $M \neq 0$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

8. Finally, $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ for $n = 1, 2, 3, \dots$

We use all of these without explanation. In fact, we can sum them up in one statement:

Direct Substitution Property If $f(x)$ is an algebraic function and $x = a$ is in the domain of $f(x)$, then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

So for a wide family of functions, the limit is the function value. Of course, the interesting problems are those when we cannot do this!

Example 2: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Example 3: $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 5x + 6}$

Note: The limit of a quotient where **both** the numerator and the denominator evaluate to 0 as $x \rightarrow a$ is called an **indeterminate form**. Sometimes we write " $\frac{0}{0}$ " (in quotes because you can't divide by zero). Another indeterminate form is " $\infty - \infty$ ". However, the algebra tricks like above (and later L'Hospital's Rule which we'll see at the end of the quarter) only work with quotients. So if we have an indeterminate difference, we try to make it into a quotient first using algebra.

Example 4: $\lim_{x \rightarrow 2} \frac{x^2 - 9}{x^2 - 6x + 9}$

Summary of algebraic trick: When you are evaluating the limit of a quotient of two **polynomials**

$$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)}$$

where **both** have $P(a) = Q(a) = 0$, then $(x - a)$ must be a factor of both. Factor, simplify the $(x - a)$ terms and reconsider the limit.

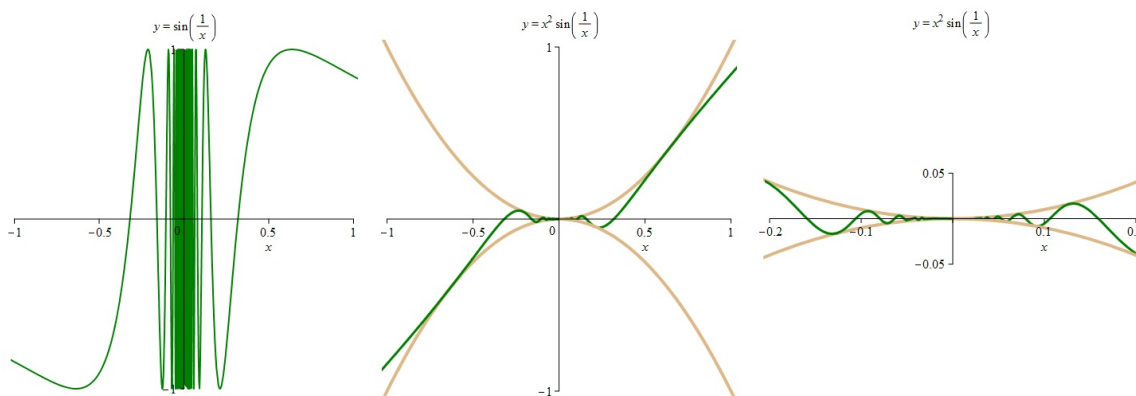
Example 5: $\lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 3} - 2}{x - 1}$

Note: This trick is called **rationalization**, where we turn the part with the square root into a polynomial so we can factor the problematic term out.

Example 6: $\lim_{x \rightarrow 3} \frac{|x - 3|}{x - 3}$

Note: Any function/formula which involves an absolute value must be considered a multi-part function and its limit computation should involve left and right limits.

Example 7: $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$



Sandwich/Squeeze/Pinching Theorem

If $g(x) \leq f(x) \leq h(x)$ (at least around $x = a$) and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.

