Section 2.2 - Limits of Functions

Last time, for the slope of the tangent problem with $y = \dfrac{2}{x}$, we had the point of tangency $(1, 2)$. We picked a second point $\left(x, \dfrac{2}{x}\right)$ and we looked at the slope of the line through these two points

$$\textbf{slope} = \frac{\frac{2}{x} - 2}{x - 1}$$

for different values of x. To find the slope of the tangent line, we made x get closer and closer to the number 1.

Today, we want to generalize this idea: instead of using the above quotient, we can have ANY formula/function to work with.

QUESTION: Given a function $f(x)$ and a number a, do the values of $f(x)$ approach a number as x approaches a?

Example 1: Let $f(x) = x^2 + 1$ and $a = 1$. As x gets close to 1, what value, if any, the numbers $f(x)$ approach?

Definition: The **limit** of $f(x)$ as x approaches a equals L (which is a number) if the values of $f(x)$ get arbitrarily close to L when x is sufficiently close to a. We write

$$\lim_{x \to a} f(x) = L.$$

So, we would write our previous answer as

$$\lim_{x \to 1} (x^2 + 1) = 2.$$

Note: The limit is NOT the same as the function value. It was for the example above with $f(x) = x^2 + 1$. It is not for the example $\frac{\frac{2}{x} - 2}{x - 1}$ as $f(1)$ is not even defined! The functions which have their function values equaling their limits are special and they get their own classification and terminology later in this chapter.

Example 2: Here are three functions. We'll look at their limits as x approaches 1:

$$f(x) = x + 1 \qquad g(x) = \frac{x^2 - 1}{x - 1} \qquad h(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

Let's start by sketching their graphs:

Now we can see their limits:

Questions:

1. Do we always have a limit?
2. If there is a limit, how do we find it?
3. What makes a limit fail to exist?

Limits for Multi-part Functions

Example 3:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

Look at what happens as x approaches 1.

Infinite Limits and Vertical Asymptotes

Example 4:

$$\lim_{x \to 0} \frac{1}{x^2}$$

Infinite limit: If the limit of a function does not exist because the values of $f(x)$ exceed every number as x approaches a, then we write

$$\lim_{x \to a} f(x) = \infty.$$

If the limit of a function does not exist because the values of $f(x)$ eventually become less than every (negative) number as x approaches a, then we write

$$\lim_{x \to a} f(x) = -\infty.$$

For example,

$$\lim_{x \to 0} 1 - \frac{1}{x^2} = -\infty.$$

Example 5:

$$\lim_{x \to 0^+} \ln x$$

Example 6:

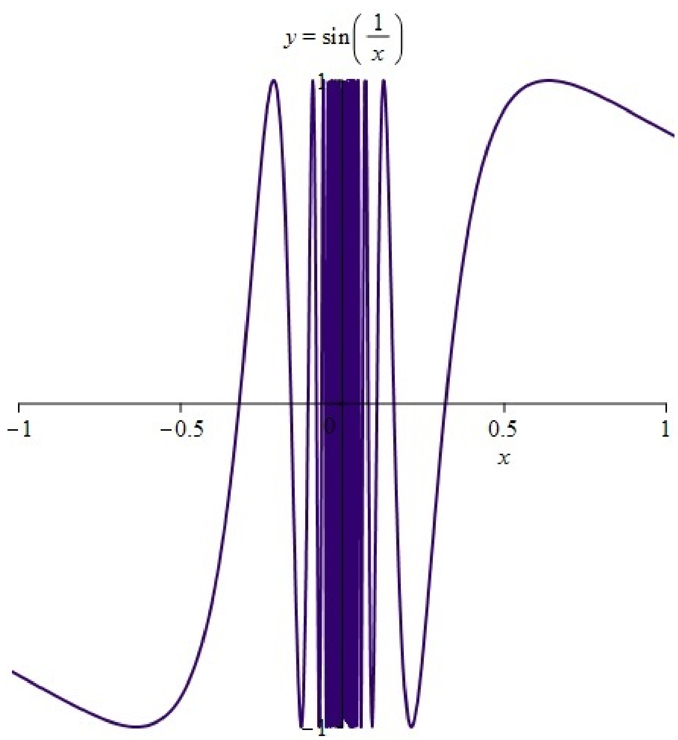
$$\lim_{x \to 2} \frac{1}{x-2}$$

The most common ways limits fail to exist happen in multi-part functions and functions with vertical asymptotes. But, other things might happen as well.

Example 7:

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

You can graph the function on a graphing calculator/app (I use Maple or Desmos) and see how it keeps oscillating when you get close to the y-axis where $x = 0$.



If the limit exists, how do we find it?

1. We can make a table of values using calculator. Eventually we have to guess the value where $f(x)$ seems to approach. Guessing is relatively easy if the answer is an integer. Can you guess the answer from the following table of values?

x	0.1	0.01	0.001	0.0001	0.00001
$f(x)$	0.807107	0.717107	0.708107	0.707207	0.707117

2. We can look at a graph. Well, you need a graph and again there may be issues with reading numbers with good precision.

3. We can use algebra and some properties of limits. We've seen that sometimes the limit is the function value. What if it is not? Well, that is the next lecture.