

Lines, Circles and Tangents to Circles

The aim of this section is to remind you of the line and circle equations and introduce the concept of a tangent which is very important in defining the derivative later on. You will also be using your algebra skills which are essential in this course and beyond.

Equations of Lines

A line is determined by a **point** (x_0, y_0) and a **slope** m .

The **point-slope equation of a line** is

$$y - y_0 = m(x - x_0)$$

We'll be writing a LOT of line equations and this is the fastest way to do it.

If you are given two points on the line, you can find its slope by computing the change in y divided by the change in x , or rise over run:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

Example 1: Find the equation of the line through the points $(2, 1)$ and $(5, 7)$ by first computing its slope.

Equations of Circles

A circle is determined by its **center** (a, b) and a **radius** r . The equation of the circle is

$$(x - a)^2 + (y - b)^2 = r^2.$$

This is says that really the distance from the point (x, y) to (a, b) is r .

Frequently we'll have a circle with center at the origin $(0, 0)$ with equation

$$x^2 + y^2 = r^2.$$

The circle of radius 1 centered at the origin is called the **unit circle** with equation

$$x^2 + y^2 = 1.$$

A circle and a line may intersect at two points, at a single point (where the line will necessarily be **tangent** to the circle), or not intersect at all.

Question: Given equations of the circle and the line, how do you determine (if any) the points of intersection and decide which one of the three pictures you have? *Without a graphing device, of course!*

Example 2: Find the intersection points, if any, of the given line with the circle and draw a sketch of the two together.

$$y = x + 5 \quad \text{and} \quad x^2 + y^2 = 9$$

Example 3: Find the intersection points, if any, of the given line with the circle and draw a sketch of the two together.

$$y = -x + 1 \quad \text{and} \quad x^2 + y^2 = 25$$

Example 4: Find the intersection points, if any, of the given line with the circle and draw a sketch of the two together.

$$3x + 4y = 25 \quad \text{and} \quad x^2 + y^2 = 25$$

All will require the use of the **Quadratic Formula:**

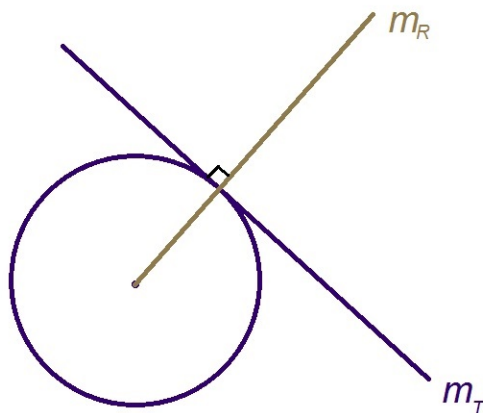
The solution(s) to the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Tangents to Circles



Recall that slopes of perpendicular lines are inverse reciprocals so they satisfy $m_1 m_2 = -1$.

Note: I added an explanation/exercise about why this is true at the end if you want to do some geometry. You don't have to remember the why. You just have to remember this fact about slopes of perpendicular lines.

The tangent line to a circle is perpendicular to the radial line drawn from the center of the circle to the point of tangency. Hence,

$$m_T m_R = -1.$$

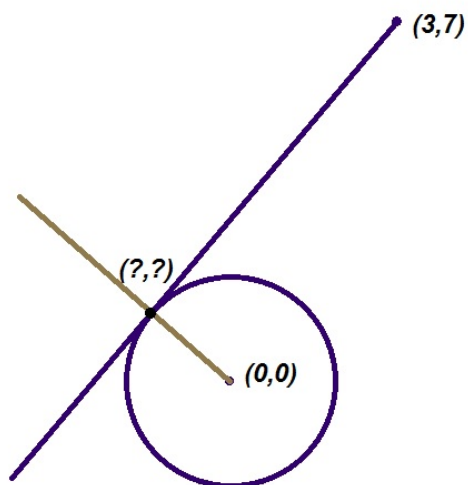
We can use this property to write down the equation of a tangent line to a circle IF we know the point of tangency.

Example 5: Find the equation of the tangent line to the circle $x^2 + y^2 = 13^2$ at the point $(12, 5)$.

Note that the point is ON the circle. How can you check that?

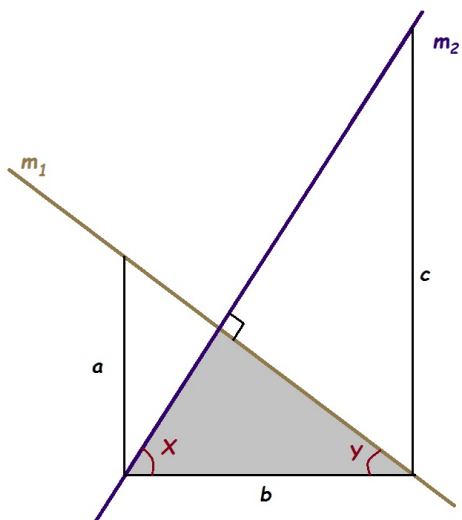
When the point given is NOT on the circle, it becomes a longer algebra problem to find the equation of the tangent line to the circle through this point. The example below is similar to a homework problem.

Example 6: Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ through the point $(3, 7)$. Note that the point is *NOT ON* the circle. How can you check that?



This type of problem where we do not know the point of tangency will show up in more problems through the course. In such problems, we call the unknown point of tangency (a, b) and try to write an equation about the slope of the tangent.

Optional Exercise: Look at the picture below and follow the steps to see why the slopes of the lines have to satisfy $m_1 m_2 = -1$.



1. The slope of the purple line is $m_2 = c/b = \tan X$. Do you see why?
2. The slope of the gold line is $m_1 = -a/b = -\tan Y$. Do you see why? Why is there a minus sign?
3. Now look at the gray triangle. How are the angles X and Y related? If you compute their tangents using the gray triangle, do you see they will be reciprocals of each other?
4. Now put it all together to verify $m_1 m_2 = -1$.