Math 124 Midterm II Review

Differentiation Techniques - Sections 3.1-3.6

1. Differentiation rules for x^a (which includes $\sqrt[n]{x}$ and $\frac{1}{x^b}$), a^x (in particular e^x), $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, $\cot x$, $\log_a x$ (in particular $\ln x$), $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$, $\cot^{-1} x$.

   The ones you have not memorized yet should go on your note sheet.
2. The sum rule, multiplication by a constant rule (we use these two from Section 3.1 without thinking), the product rule, the quotient rule, the **chain rule**.
3. Implicit Differentiation - first and second derivatives - Section 3.5.
4. Logarithmic Differentiation - Section 3.6
5. Computing $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for parametric curves - Section 10.2.

For the first four, besides the problems in the relevant sections and exam archives, you can look at Problems 1-42 in Chapter 3 Review. For second derivatives of implicit functions, look at Problems 35-40 in Section 3.5. For parametric curves, look at problems 1-6, 11-20 in Section 10.2.

Equations of Tangent Lines and Linear Approximation - Section 3.10

Regardless of how a curve is described, explicitly (for example, $y = x\sin x$), implicitly (for example, $xy + \sin y = y^2 - x$) or using parametric equations (for example, $x = t^2 + t, y = t^3 + 7$), the slope of the tangent line at a point is always the value of the derivative of y with respect to x at that point, which we denote by $f'(x)$, y' or $\frac{dy}{dx}$. This is because the slope of a line on the xy-plane is always $\frac{\Delta x}{\Delta y} = \frac{\text{rise}}{\text{run}}$.

For explicit or implicit functions, we can use the equation of the tangent line for an approximation of the functions themselves. You basically replace the (complicated) original equation with the (simple) tangent line equation, changing the equality $=$ to an approximation $\approx$. See problems 1-10, 23-28 (be careful in 28, you have to convert to radians) for examples with explicit functions. See the exam archives (look at the final exams, too) for examples with implicit functions.

If you have variables other than x and y in a linear approximation problem, decide which one is independent (like x) and which one is dependent (like y). This choice will help you write the tangent line equation with slope equal to the value of $\frac{dy}{dx}$. Which one you choose to be which will not change the final answer. It will effect the way you do your (implicit) differentiation.

For example, if the equation is

$$\sin\left(\frac{\pi}{4}\theta\right) = z^2$$

and the point is $\theta = 2$ and $z = 1$, we can think of θ as a function of z (so z like x and θ like y) then the implicit differentiation will be

$$\frac{\pi}{4}\cos\left(\frac{\pi}{4}\theta\right)\frac{d\theta}{dz} = 2z,$$

the slope computed by

$$\frac{\pi}{4}\cos\left(\frac{\pi}{4}2\right)\frac{d\theta}{dz} = 2 \qquad \text{so} \qquad \frac{d\theta}{dz} = \frac{8}{\pi}$$

and the line equation given by

$$\theta - 2 = \frac{8}{\pi}(z-1).$$

On the other hand, if you think of z as a function of θ (so θ like x and z like y) then the implicit differentiation will be

$$\frac{\pi}{4}\cos\left(\frac{\pi}{4}\theta\right) = 2z\frac{dz}{d\theta},$$

the slope computed by

$$\frac{\pi}{4}\cos\left(\frac{\pi}{4}2\right) = 2\frac{dz}{d\theta} \qquad \text{so} \qquad \frac{dz}{d\theta} = \frac{\pi}{8}$$

and the line equation given by

$$z - 1 = \frac{\pi}{8}(\theta - 2).$$

The two line equations and the linear approximations $\theta - 2 \approx \frac{8}{\pi}(z-1)$ or $z - 1 \approx \frac{\pi}{8}(\theta - 2)$ are the same.

Related Rate Problems - Section 3.9

In a related rate problem, you have two (or more) quantities that are changing with *time*. You first relate those quantities by finding an equation containing them. Then, you differentiate the equation *with respect to time*, keeping in mind that all the variables in your equation are really functions of time. You will use the chain rule. Then, you plug in everything you know, except for the missing rate.

In these problems, it is better to use the quotient notation like $\frac{dx}{dt}$, $\frac{dh}{dt}$, $\frac{d\theta}{dt}$, etc. The units in the problem can help you here. For example, if you have meters per second, you know a distance quantity must be changing. If you have something turning or rotation, an angle must be changing. You want to write the rate in radians per time unit.

For practice, you can look at more problems from the textbook or the exam archives. There are related rate problems in second midterms and final exams. Below are hints for some of the related rate problems from my second midterms.

1. (Winter 2007) Volume of a cone is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is its height.
2. (Winter 2008) The hour hand completes one revolution every 12 hours, the minute hand completes one revolution in 1 hour. 1 revolution is 2π radians.
3. (Autumn 2009) Pythagorean Theorem.
4. (Spring 2011) Divide the trapezoid into two triangles and a rectangle and use similar triangles.
5. (Autumn 2011) Volume of a cone is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is its height. Volume of a cylinder is $V = \pi r^2 h$ where r is the radius of the base and h is its height. Use similar triangles to find the volume of the water in the conical tank in terms of its height.
6. (Winter 2013) To get the area on the right picture in terms of the vertical distance, divide up the hexagon into triangles and a rectangle. You will use the Pythagorean Theorem at some point.
7. (Spring 2013) Form a right triangle in the picture and use one of the trigonometric functions.

Applications of derivatives - Section 4.1

Critical points, finding absolute minimum and absolute maximum for continuous functions on intervals of the form $[a, b]$.