Math 112, Spring 2019, Solutions to Midterm I

1. (a) $f'(s) = D$, $\dfrac{f(s+r)-f(s)}{r} = C$, $\dfrac{f(s)-f(r)}{s-r} = \text{none}$, $f(s+r)-f(s) = A$

   (b) Given that $f(s+r)-f(s) = \dfrac{2sr+r}{(s+3)(s+r)}$

   (i) (2 points) Find the average rate of change of $f(x)$ from $x=3$ to $x=5$.

$$\frac{f(s+r)-f(s)}{r} = \frac{2s+1}{(s+3)(s+r)}$$

$$\frac{f(5)-f(3)}{5-3} = \frac{2\cdot 3+1}{(3+3)(3+2)} = \frac{7}{30}$$

   (ii) (3 points) Compute $f'(7)$.

$$\frac{f(x+h)-f(x)}{h} = \frac{2x+1}{(x+3)(x+h)}$$

when $h=0$

$$f'(x) = \frac{2x+1}{(x+3)(x)}$$

so

$$f'(7) = \frac{2\cdot 7+1}{(7+3)(7)} = \frac{15}{70} = \frac{3}{14}$$

2. Differentiate the following functions.

   (a) (3 points) $f(x) = \dfrac{x^3}{7} - 2\sqrt{x} + \dfrac{4}{x}$. Your answers should not have negative exponents.

$$f'(x) = \frac{3}{7}x^2 - \frac{1}{\sqrt{x}} - \frac{4}{x^2} = \frac{3}{7}x^2 - \frac{1}{x^{1/2}} - \frac{4}{x^2}$$

   (b) (3 points) $f(x) = (7x^2-5)\sqrt{3x+1}$. You do not have to simplify your answer.

$$f'(x) = 14x\sqrt{3x+1} + \frac{(7x^2-5)\cdot 3}{2\sqrt{3x+1}}$$

   (c) (5 points) $f(x) = \dfrac{(x-2)^2}{(x^2+1)^3}$. Simplify your answer and find the values of x where the graph of $y = f(x)$ has a horizontal tangent line.

$$f'(x) = \frac{2(x-2)(x^2+1)^3 - (x-2)^2\cdot 3\cdot (x^2+1)^2\cdot 2x}{(x^2+1)^6}$$

$$f'(x) = \frac{2(x-2)(x^2+1)^2\left[x^2+1-3x(x-2)\right]}{(x^2+1)^6}$$

$$f'(x) = \frac{2(x-2)\left[-2x^2+6x+1\right]}{(x^2+1)^4} = 0$$

when $x=2$ or

$$x = \frac{-6\pm\sqrt{36-4(-2)}}{-4} = \frac{3\pm\sqrt{11}}{2}$$

3. (a) (1 point) Estimate the average rate of change of altitude for Balloon B during the 0.1 second interval starting at $t = 10$.

$$\approx B'(10) \approx -1.3$$

(b) (1 point) When is the distance between them maximum?

$$t = 5$$

(c) (1 point) When will Balloon B reach its highest altitude?

$$t = 4$$

(d) (3 points) Find the instantaneous rate of change of altitude of Balloon A at $t = 4.34$ seconds. Be as precise as you can. Is it going up or down?
Line equation for A' graph:

$$y = \frac{2}{7}x - 2$$

so $A'(t) = \frac{2}{7}t - 2$. Then

$$A'\left(\frac{4.34}{60}\right) = \frac{2}{7}\left(\frac{4.34}{60}\right) - 2 = -\frac{59.38}{30}$$

(The dividing by 60 is for the unit conversion.) The balloon is going down because the derivative is negative.

(e) (1 point) Find the interval where both balloons are going down towards the ground.

$$4 < t < 7$$

(f) (2 points) Describe the motion of Balloon A as going up/going down, speeding up/slowing down in the interval $[0, 7]$.
Going down, slowing down.

4. (a) (2 points) Approximate the cost of producing the 501st Top.

$$MC(x) = C'(x) = 1.2x$$

$$MC(5) = 1.2 \cdot 5 = 6 \text{ dollars per Top}$$

(b) (3 points) At what quantity is Marginal Revenue minimum?

$$R(x) = (x^2 - 5.75x + 9.5)x = x^3 - 5.75x^2 + 9.5x$$

$$MR(x) = R'(x) = 3x^2 - 11.5x + 9.5 \text{ has minimum value when } MR'(x) = 0$$

so

$$6x - 11.5 = 0 \text{ or } x = 11.5/6 \text{ hundred Tops}$$

(c) (3 points) Do you make a profit from the sale of the 201st Top?

$$MP(2) = MR(2) - MC(2) = -1.9 \text{ so no profit (a loss) from the 201st Top}$$

(d) (3 points) Find the maximum profit.

$$MP(x) = MR(x) - MC(x) = 3x^2 - 12.7x + 9.5 = 0$$

when

$$x = \frac{12.7 \pm \sqrt{12.7^2 - 4 \cdot 3 \cdot 9.5}}{6} \approx 0.97 \text{ or } 3.26$$

Where doe MP switch from + to $-$?
MP graph is a parabola opening up so it switches from + to - at its first root, $x = 0.97$ hundred Tops. Maximum Profit is

$$P(0.97) = R(0.97) - C(0.97) = 3.15 \text{ hundred dollars}$$