Rules of Differentiation

First, we need to know how to differentiate the "pieces" making up a formula:

$$\boxed{\frac{d}{dx}x^m = mx^{m-1}, \qquad \frac{d}{dx}\ln x = \frac{1}{x}, \qquad \frac{d}{dx}e^x = e^x \quad .}$$

Then we have the rule for the coefficients and sums: we leave them alone. Plus signs, minus signs, coefficients in front of numbers stay where they are. Then, we have the Product and Quotient Rules:

$$\boxed{(fg)' = f'g + fg' \quad \text{and} \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

And last, but not least, we have the Chain Rule:

$$\boxed{\frac{d}{dx}f(g(x) = f'(g(x))g'(x)}$$

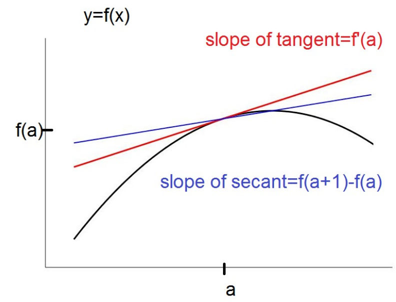
If we use the Chain Rule when the outside function $f(x)$ is a power function, exponential, or the logarithm, we get the following formulas:

$$\boxed{\frac{d}{dx}\left(g(x)\right)^m = m(g(x))^{m-1}g'(x) \qquad \frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x) \qquad \frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}}$$

There are many differentiation problems in the Midterm 2 Archives. Do enough until you feel comfortable with differentiation. Usually we do not ask you to simplify the derivatives, but make sure you use parentheses correctly. If you have to simplify a derivative, keep in mind you want to factor as much as you can.

Interpreting the Derivative and the Second Derivative

We use the derivative in three main ways:



I. Geometrically the value of the derivative gives the slope of the tangent line to the graph of f.

II. The value of the derivative gives the instantaneous rate of change of the function. For example, the derivative of an altitude function gives the (vertical) velocity of a balloon.

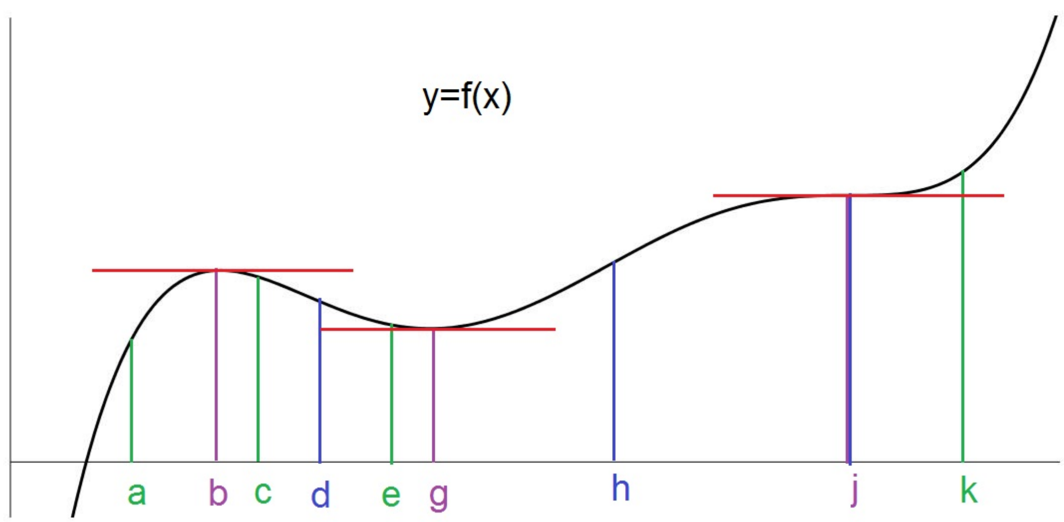
III. We can use the derivative to approximate change as in

$$f'(a) \approx f(a+1) - f(a)$$

As a consequence of I. above, we get the following:

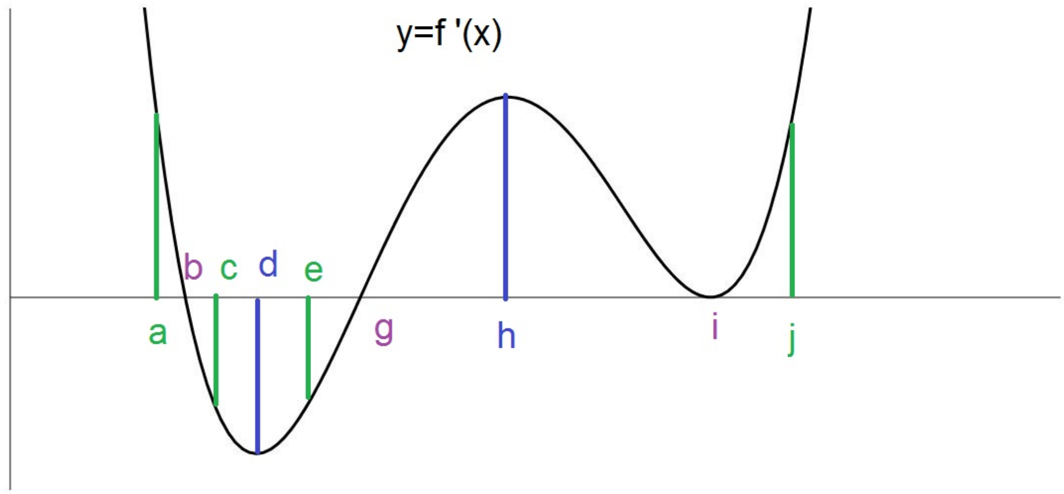
1. If $f'(a) > 0$, the function is increasing at $x = a$ (at the rate of $f'(a)$). If $f'(a) < 0$, the function is decreasing at $x = a$ (at the rate of $f'(a)$)
2. If $f'(x) = 0$, the graph of the function has a horizontal tangent at $x = a$. This could be a local max, local min, or a horizontal inflection. We can distinguish which one by looking at how $f'(x)$ changes at $x = a$:
   (a) If $f'(x)$ switches from positive to negative, $f(a)$ is a local max.
   (b) If $f'(x)$ switches from negative to positive, $f(a)$ is a local max.
   (c) If $f'(x)$ does not switch sign, $(a, f(a))$ is a horizontal inflection point.
3. If $f'(x)$ is increasing we have a concave up graph. In this case, $f'' > 0$, since f'' is the derivative of f'. If $f'(x)$ is decreasing we have a concave down graph. In this case, $f'' < 0$.

Below is the graph of $f(x)$ as an example:



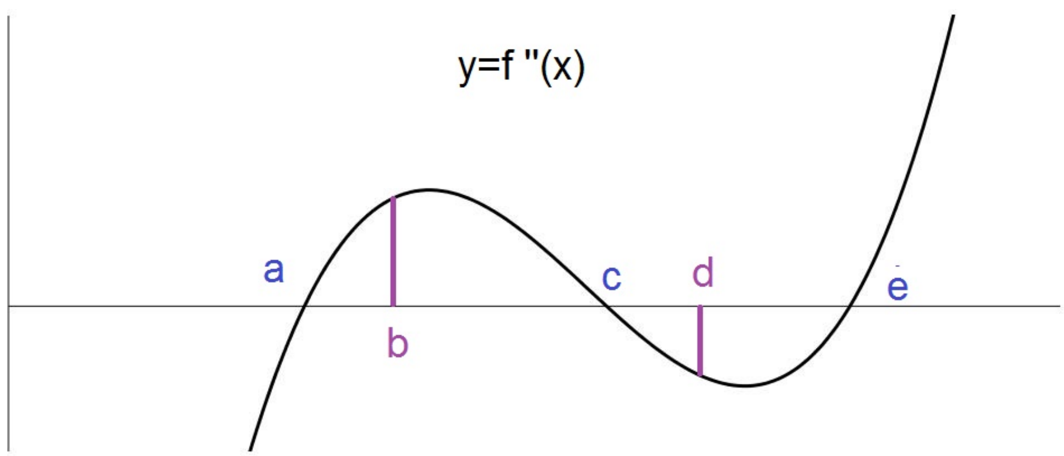
At points b, g and j, the graph of $f(x)$ has a horizontal tangent line so $f'(b) = f'(g) = f'(j) = 0$. All are critical points. More specifically, $f(b)$ is a local maximum value, $f(g)$ is a local minimum value and $(k, f(k))$ is a horizontal inflection point.
At d, h and j the function $f(x)$ has an inflection point. So $f''(d) = f''(h) = f''(j) = 0$.
For the other points we can see:
At a, the function is increasing and the graph of $f(x)$ is concave down so: $f'(a) > 0$ and $f''(a) < 0$.
At c, the function is decreasing and the graph of $f(x)$ is concave down so: $f'(c) < 0$ and $f''(c) < 0$.
At e, the function is decreasing and the graph of $f(x)$ is concave up so: $f'(e) < 0$ and $f''(e) > 0$.
At k, the function is increasing and the graph of $f(x)$ is concave up so: $f'(k) > 0$ and $f''(k) > 0$.

We can get information about the function $f(x)$ from its derivative $f'(x)$:



At b, g and i, the function $f(x)$ has a critical point and a horizontal tangent because $f'(b) = f'(g) = f'(i) = 0$. We can see from the switch of sign of $f'(x)$ that $x = b$ gives a local max and $x = g$ gives a local min for $f(x)$. At $x = i$ the function $f(x)$ has a horizontal inflection because although $f'(i) = 0$, it does not change sign.
At d, h and i, the function $f(x)$ has an inflection point because $f'(x)$ changes direction there (so $f''(d) = f''(h) = f''(i) = 0$)
At a, the function is increasing and the graph of $f(x)$ is concave down because $f'(a) > 0$ and $f''(a) < 0$.
At c, the function is decreasing and the graph of $f(x)$ is concave down because $f'(c) < 0$ and $f''(c) < 0$. At e, the function is decreasing and the graph of $f(x)$ is concave up because $f'(e) < 0$ and $f''(e) > 0$.
At j, the function is increasing and the graph of $f(x)$ is concave up because $f'(k) > 0$ and $f''(k) > 0$.

We can also get (limited) information about the function $f(x)$ from its second derivative $f''(x)$:



At a, c and e, the function $f(x)$ has inflection points because $f''(a) = f''(c) = f''(e) = 0$ and f'' changes sign each time forcing $f(x)$ to change its concavity.
At b the graph of f is concave up because $f''(b) > 0$ and at d the graph of f is concave down because $f''(d) < 0$.

Application: Optimization - Finding the Maximum or Minimum Values of Functions

Since maximum or minimum happens when f' changes sign, we first look to see where $f' = 0$ (it has to pass through zero to change its sign.) If we are are working with a formula, first we solve $f'(x) = 0$. Any solution to $f;(x) = 0$ is called a **critical value**. Then, we distinguish two cases:

CASE 1 The domain is specified as $a \leq x \leq b$. In that case we compare $f(a)$, $f(b)$ and $f(\text{critical value})$. We pick the max or min as desired. For example, we may be asked to maximize Profit when the quantity is restricted between 50 and 100 Things.

CASE 2 The domain is not of the form above. (This is more common. For example in Cost-Revenue-Profit problems we have $q > 0$.) In this case to determine or verify whether the critical number gives a minimum or maximum for $f(x)$ we look for EITHER the change of sing of $f'(x)$ at the critical value, by checking f' before and after the critical value OR we look at the value of the second derivative and concavity at the critical value.