

The Definite Integral and Applications

The Definite Integral gives a number. We use integration rules to find the integral, omit the $+C$ (even if we put it it will cancel out so we don't bother) and evaluate the integral at the top and bottom limits and take the difference.

$$\int_0^3 2x + 5 dx = x^2 + 5x \Big|_0^3 = (3^2 + 5 \cdot 3) - (0^2 + 5 \cdot 0) = 24 - 0 = 24$$

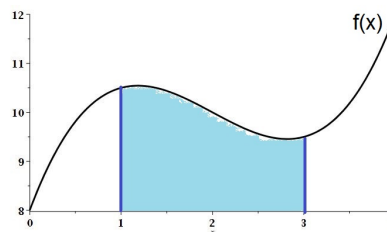
Application: Computing Areas Using Integrals

If $f(x) > 0$ and the graph is above the x -axis, the integral gives the area under the curve and above the x -axis. The graph on the right is of

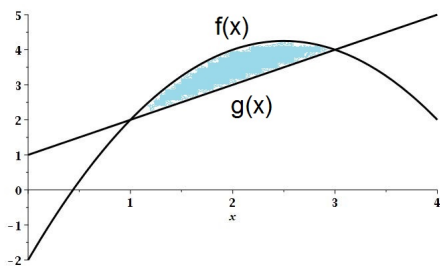
$$f(x) = 0.5x^3 - 3x^2 + 5x + 8$$

so the area shown is given by the definite integral

$$\int_1^3 0.5x^3 - 3x^2 + 5x + 8 dx.$$



If the function $f(x) < 0$, the integral is negative of the area.



If the area is between two graphs, we integrate "top curve minus bottom curve". On the right are the graphs of $f(x) = -x^2 + 5x - 2$ and $g(x) = x + 1$. The area between them is given by the integral

$$\int_1^3 (-x^2 + 5x - 2) - (x + 1) dx$$

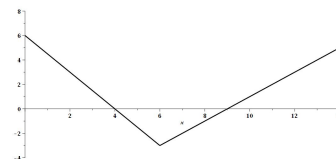
If you cannot see clearly the x -values for the intersection points from the graph, you can solve for them by setting the two equations equal to each other. In this case

$$-x^2 + 5x - 2 = x + 1.$$

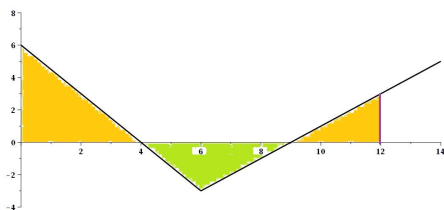
Application: Using Areas to Compute Integrals

To find $\int_0^{12} f(x) dx$ for the function whose graph is given, we could work out the equations of the two lines making up $f(x)$ using the points on the graph. If we did, the answer would look like this:

$$\int_0^{12} f(x) dx = \int_0^6 -\frac{3}{2}x + 6 dx + \int_6^{12} \frac{5}{6}x - 5 dx$$



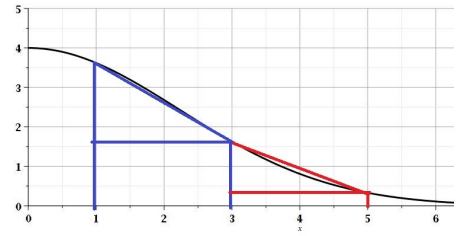
The integral is split because the formula changes at $x = 6$. You can see why this method is long and not practical.



It is better to interpret the integral as area. Since areas under the x -axis give negative integrals, we add the blue areas and subtract the green area (all triangles) to calculate the integral:

$$\int_0^{12} f(x) dx = \frac{4 \times 6}{2} - \frac{5 \times 2.5}{2} + \frac{3 \times 2.5}{2}$$

If we are given the graph of the function instead of its formula, we can approximate the integral by approximating the area. To approximate area under a curve, draw some rectangles and triangles very close to the actual curve and compute those areas.



Application: The Definite Integral of a Rate of Change Gives the Change

This is the most important application of integration for us. In math notation it says

$$\int_a^b f'(x) dx = f(b) - f(a)$$

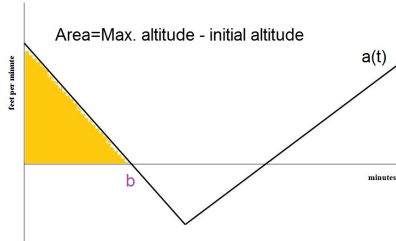
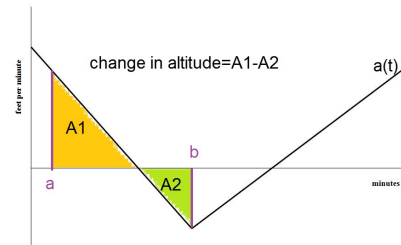
We integrate rate of change to get total change.

Change of Altitude from Rate of Ascent

For example, if $a(t) = A'(t)$ is rate of ascent of a balloon - so it is the rate of change of its height $A(t)$

$$\int_a^b a(t) dt = \int_a^b A'(t) dt = A(b) - A(a) \quad \text{is the change in altitude.}$$

Areas under the t axis will make a negative contribution to the integral.



At the point where $a(t)$ switches from positive to negative, $t = b$ in picture, the altitude makes a maximum value.

$$\int_0^b a(t) dt = A(b) - A(0) = \text{Maximum Altitude} - \text{Initial Altitude}$$

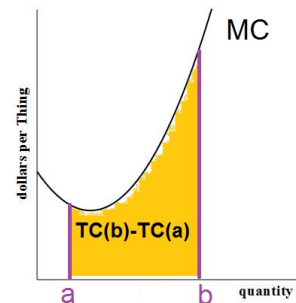
Change in Cost and Revenue from Marginal Cost and Marginal Revenue

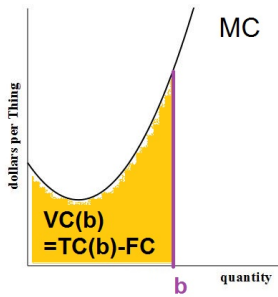
If we integrate Marginal Cost, we get the change in Total Cost:

$$\int_a^b MC(q) dq = TC(b) - TC(a).$$

We also get the change in Variable Cost because MC is also the derivative of Variable Cost:

$$\int_a^b MC(q) dq = VC(b) - VC(a)$$





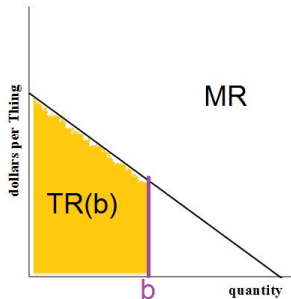
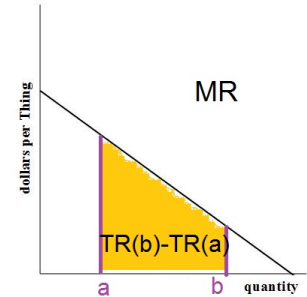
In particular, since $VC(0) = 0$

$$\int_0^b MC(q) dq = VC(b) - VC(0) = VC(b)$$

So we can use an integral of MC to compute VC at any quantity.

If we integrate Marginal Revenue, we get the difference in Total Revenue

$$\int_a^b MC(q) dq = TR(b) - TR(a)$$



In particular, since $TR(0) = 0$

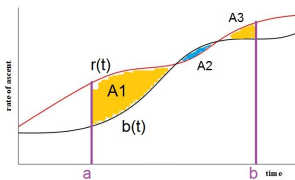
$$\int_0^b MR(q) dq = TR(b) - TR(0) = TR(b)$$

So we can use an integral of MR to compute TR at any quantity.

Application: Working with Two Rates

The integral of a difference can be interpreted as area between two functions. Since $(f - g)' = f' - g'$, integrating a difference of two rates gives the change of their difference.

Two Rates of Ascent

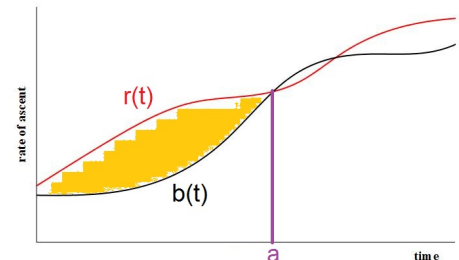


On the left are graphs of rates of ascent $r(t)$ and $b(t)$ for two balloons, Red and Black. Initially they are at the same altitude. When $r(t) < b(t)$, the Red balloon will rise faster increasing the distance between them. When $r(t) > b(t)$, the Black balloon will start to catch up decreasing the distance between them.

$$\int_a^b r(t) - b(t) dt = \text{change in the distance between the balloons} = A1 - A2 + A3$$

We know maximum distance is attained when we switch from $r(t) > b(t)$ to $r(t) < b(t)$. Say it happens at $t = a$. If they initially start at the same height, then,

$$\int_0^a r(t) - b(t) dt = \text{Maximum Distance between them} = \text{area.}$$

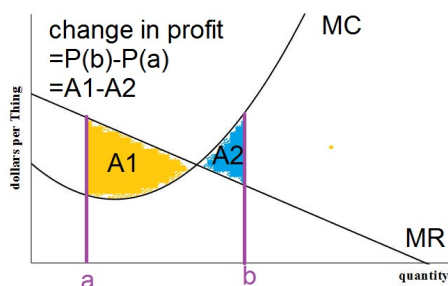
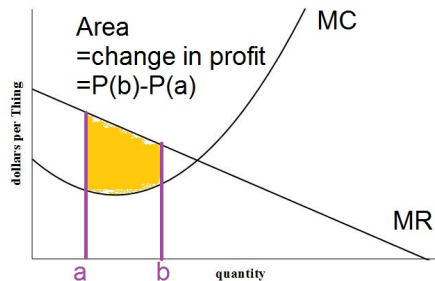


Marginal Revenue and Marginal Cost Together

Since $P(q) = TR(q) - TC(q)$ we have $P'(q) = MR(q) - MC(q)$
so

$$\int_a^b MR(q) - MC(q) dq = P(b) - P(a) = \text{change in Profit,}$$

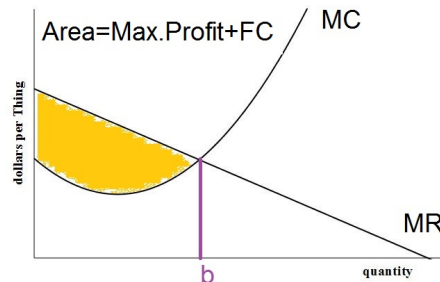
When we have the graphs of MR and MC we can interpret this as the area between two functions.



When $MR < MC$, profits start decreasing. So in the integral, area between the two will make a negative contribution to the change in Profit.

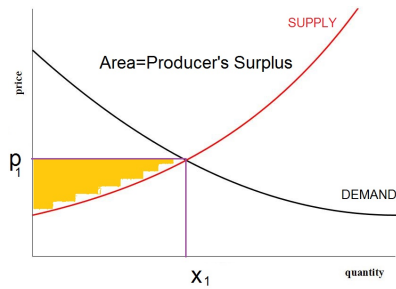
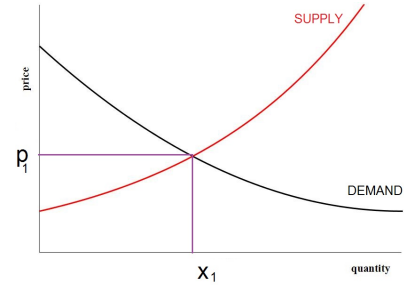
We know **maximum profit** is attained when we switch from $MR > MC$ to $MR < MC$. This is exactly the place where we switch from $P' > 0$ to $P' < 0$. (Why?) Say it happens at $q = b$. Then,

$$\begin{aligned} \int_0^b MR(q) - MC(q) &= P(b) - P(0) \\ &= P(b) - (-FC) = \text{Maximum Profit} + FC. \end{aligned}$$



Application: Producer's and Consumer's Surplus

Recall the basic facts about Supply and Demand: The Supply function is increasing. When prices go up, so does the quantity supplied. The Demand function is decreasing. When prices go up, the quantity decreases as fewer people are willing to buy. Also, The quantity (q or x) is always on the x -axis, and price p is on the y -axis. Their intersection point (x_1, p_1) , which you can find by setting the demand and supply equations equal to each other, is called the **equilibrium point**.



The Producer's Surplus is given by

$$p_1 x_1 - \int_0^{x_1} \text{"Supply Function"} dx.$$

It is the shaded area on the left.

The Consumer's Surplus is given by

$$\int_0^{x_1} \text{"Demand Function"} dx - p_1 x_1$$

It is the shaded area on the right.

