Math 124, Winter 2022 Solutions to Midterm II

1. (a) $f'(x) = 6xe^{x^2} + \dfrac{21x^2}{1+x^3} - \dfrac{\sec(6+\sqrt{x})\cdot\tan(6+\sqrt{x})}{2\sqrt{x}}.$

   (b) $f'(x) = \dfrac{12\sin^2(4x)\cos(4x)}{\sin^3(4x)+5}$

   (c) Logarithmic differentiation:

$$y = (1+x^2)^{\sqrt{x}}$$
$$\ln y = \ln\left((1+x^2)^{\sqrt{x}}\right) = \sqrt{x}\ln(1+x^2)$$
$$\frac{y'}{y} = \frac{\ln(1+x^2)}{2\sqrt{x}} + \frac{2x\sqrt{x}}{1+x^2}$$
$$y' = \left(\frac{\ln(1+x^2)}{2\sqrt{x}} + \frac{2x\sqrt{x}}{1+x^2}\right)(1+x^2)^{\sqrt{x}}$$

2. (a) Differentiate to get $26x - 16xy^2 - 16x^2yy' + 27y^2y' = 0$ and

$$y' = \frac{16xy^2 - 26x}{27y^2 - 16x^2y}.$$

   (b) When $x = 1$ and $y = 2$

$$26 - 16\cdot 4 - 16\cdot 2y' + 27\cdot 4y' = 0$$

   so

$$y' = \frac{64-26}{108-32} = \frac{38}{76} = \frac{1}{2}.$$

   Therefore, the tangent line is given by

$$y - 2 = \frac{1}{2}(x-1).$$

   (c) From

$$b - 2 \approx \frac{1}{2}(1.1-1).$$

   we get $b \approx 2.05$.

   (d) Differentiating $26x - 16xy^2 - 16x^2yy' + 27y^2y' = 0$. again

$$26 - 16y^2 - 32xyy' - 32xyy' - 16x^2y'y' - 16x^2yy'' + 54yy'y' + 27y^2y'' = 0$$

   When $x = 1$, $y = 2$, and $y' = 1/2$ we get

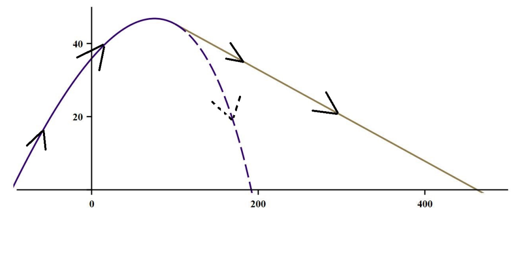
$$26 - 64 - 32 - 32 - 4 - 32y'' + 27 + 108y'' = 0$$

   so $y'' = 79/76$.

3. (a) Towards right.

   (b) Solve

$$0 = -\frac{t^3}{3} - 15t^2 + 54t = t\left(-\frac{t^2}{3} - 15t + 54\right)$$



   to get $t = 0$ (which is the starting time) or

$$t = \frac{15 \pm \sqrt{225 + \frac{54\cdot 4}{3}}}{-2/3} = \frac{-45 \pm 9\sqrt{33}}{2}.$$

   The time when it would have hit the x-axis is

$$t = \frac{-45 + 9\sqrt{33}}{2} \approx 3.35.$$

(c) First, to find the tangent line equation

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-t^2 - 30t + 54}{-18t + 116}$$

so the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{t=2} = -\frac{1}{8}.$$

Since $x(2) = -36 + 232 - 96 = 100$ and $y(2) = -\frac{8}{3} - 60 + 108 = \frac{136}{3}$ the tangent line is given by

$$y - \frac{136}{3} = -\frac{1}{8}(x - 100).$$

The particle hits the x-axis when $y = 0$ so at the point with

$$x = \frac{138 \cdot 8}{3} + 100 = \frac{1388}{3} \approx 462.7.$$

(d) We need a parametric equation for its y-coordinate to set equal to 0. We have $y'(2) = -10$ and $y(2) = \frac{136}{3}$ so for $t \geq 2$,

$$y(t) = -10(t - 2) + \frac{136}{3} = 0$$

when

$$t = \frac{136}{30} + 2 = \frac{196}{30}.$$

4. (a) We are given $\frac{d\theta}{dt} = -5$ radians per second so we relate h and θ:

$$h = 20 - 13\cos\theta.$$

Differentiating both sides with respect to t:

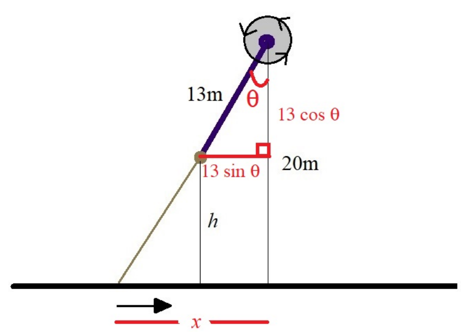
$$\frac{dh}{dt} = -13\sin\theta \cdot \frac{d\theta}{dt}.$$

When $h = 15$, $\cos\theta = 5/13$, so

$$\sin\theta = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13}.$$

So,

$$\frac{dh}{dt} = -13 \cdot \frac{12}{13} \cdot (-5) = -60 \text{ meters per minute.}$$



(b) This time to find dx/dt, we relate θ and x:

$$\tan\theta = \frac{x}{20}$$

so

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}.$$

Since $\sec\theta = 1/\cos\theta$ we have

$$\left(\frac{13}{5}\right)^2 \cdot (-5) = \frac{1}{20} \cdot \frac{dx}{dt}.$$

so

$$\frac{dx}{dt} = -676 \text{ meters per minute.}$$