Solutions to Midterm I

1. (a) $f'(x) = 6x + \dfrac{1}{2x^{3/2}} + 5e^x$.

   (b) $g'(t) = \dfrac{6t(t\cos t + \sin t) - (3t^2 - 1)(\cos t - t\sin t + \cos t)}{(t\cos t + \sin t)^2}$.

   (c) Use the *limit definition of the derivative* to compute .

$$\frac{d}{dx}\sqrt{x+3} = \lim_{h\to 0}\frac{\sqrt{x+h+3}-\sqrt{x+3}}{h}\cdot\frac{\sqrt{x+h+3}+\sqrt{x+3}}{\sqrt{x+h+3}+\sqrt{x+3}} = \lim_{h\to 0}\frac{x+h+3-(x+3)}{h\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h+3}+\sqrt{x+3}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h+3}+\sqrt{x+3}} = \frac{1}{\sqrt{x+3}+\sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}$$

   (d) A graph of $y = f(x)$ with domain $(-2, 9.5)$ is given on the right.

   (i) Note that $f'(1) = 0$, $f'(8) = -20$, $f'(3) > f'(8)$ because it is less steep and negative, and $f'(-1) > f'(0)$ because it is steeper and positive. So

$$f'(8) < f'(3) < f'(1) < f'(0) < f'(-1)$$

   (ii) At $x = 6$ where there is a "corner" $\lim_{h\to 0^+}\dfrac{f(x+h)-f(x)}{h} \neq \lim_{h\to 0^-}\dfrac{f(x+h)-f(x)}{h}$.

2. (a) $\lim_{x\to 0} x^2\sin\left(\dfrac{1}{x}\right) = 0$ by the Sandwich/Squeeze theorem as

$$-|x|^2 \le x^2\sin\left(\frac{1}{x}\right) \le |x|^2$$

   and $\lim_{x\to 0}|x|^2 = 0$.

   (b) $\lim_{t\to\infty}\dfrac{3+t^2}{5t^2+\sqrt{1+9t^4}} = \lim_{t\to\infty}\dfrac{\frac{3+t^2}{t^2}}{\frac{5t^2+\sqrt{1+9t^4}}{t^2}} = \lim_{t\to\infty}\dfrac{\frac{3}{t^2}+1}{5+\frac{\sqrt{1+9t^4}}{\sqrt{t^4}}} = \lim_{t\to\infty}\dfrac{\frac{3}{t^2}+1}{5+\sqrt{\frac{1}{t^4}+9}} = \dfrac{1}{8}$

   (c) $\lim_{h\to 0}\left(\dfrac{1}{h} - \dfrac{1}{h^2+h}\right) = \lim_{h\to 0}\left(\dfrac{h+1}{h(h+1)} - \dfrac{1}{h^2+h}\right) = \lim_{h\to 0}\left(\dfrac{h+1-1}{h(h+1)}\right) = \lim_{h\to 0}\dfrac{1}{h+1} = 1$

   (d) When $x = 2$, the denominator $(x-1)(x-2)$ is 0. If the numerator were not zero, this limit would have been ∞, $-\infty$ or DNE. So to get the limit to be 7, we must have the numerator evaluate to 0 at $x = 2$:

$$2^2 + 2a - 10 = 0$$

   so $a = 3$. Now, we compute the limit and make sure it is 7 (and not some other number):

$$\lim_{x\to 2}\frac{x^2+3x-10}{x^2-3x+2} = \lim_{x\to 2}\frac{(x-2)(x+5)}{(x-1)(x-2)} = \lim_{x\to 2}\frac{x+5}{x-1} = 7$$

3. (a) $\dfrac{h(2)-h(0)}{2-0} = \dfrac{-12.5\cdot 4 + 80\cdot 2 + 12 - 12}{2} = \dfrac{110}{2} = 55$ meters per second.

   (b) The derivative $h'(t) = -25t + 80$ so $h'(2) = -50 + 80 = 30$ meters per second. It is going up because the height is increasing.

   (c) When the velocity is -30 meters per second:

$$-30 = -25t + 80$$

   so at $t = 110/25 = 4.4$ seconds.

   (d) $h''(t) = -25$ meters per second squared.

(e) It momentarily stops before falling down, so 0.

(f) When $-25t + 80 = 0$, $t = 80/25 = 3.2$ seconds. Then, $h(3.2) = 140$ meters.

4. The slopes of the tangent lines must be $-7/9$ because they are perpendicular to the given line with slope $9/7$. Since

$$y' = \frac{5(2x-1) - (5x+1)\cdot 2}{(2x-1)^2} = -\frac{7}{(2x-1)^2}$$

at the point of tangency $(a, y(a))$ we must have

$$-\frac{7}{9} = \frac{7}{(2a-1)^2}$$

so $2a - 1 = \pm 3$ giving:

At $a = 2$, $y(2) = 11/3$ and the equation of the line is

$$y - \frac{11}{3} = -\frac{7}{9}(x-2).$$

At $a = -1$, $y(-1) = 4/3$ and the equation of the line is

$$y - \frac{4}{3} = -\frac{7}{9}(x+1).$$