Math 124, Spring 2022, Solutions to Midterm II

1. (a)

$$y' = \cos\left(4^x + \sqrt{1-e^{2x}}\right)\left(\ln 4 \cdot 4^x - \frac{2e^{2x}}{2\sqrt{1-e^{2x}}}\right)$$

(b)

$$\ln y = \ln\left(\left(x^2+5\right)^{\ln x}\right) = (\ln x)\ln\left(x^2+5\right)$$

$$\frac{y'}{y} = \frac{1}{x}\ln\left(x^2+5\right) + (\ln x)\frac{2x}{x^2+5}$$

$$y' = \left(\frac{\ln\left(x^2+5\right)}{x} + \frac{2x\ln x}{x^2+5}\right)\left(x^2+5\right)^{\ln x}$$

(c) Implicit differentiation:

$$5(3x^2+4y^2)^4(6x+8yy') = 12x - 14yy'$$

Now solve for y':

$$30x(3x^2+4y^2)^4 + 40yy'(3x^2+4y^2)^4 = 12x - 14yy'$$

$$30x(3x^2+4y^2)^4 - 12x = -14yy' - 40yy'(3x^2+4y^2)^4 = -(14y + 40y(3x^2+4y^2)^4)y'$$

$$y' = -\frac{30x(3x^2+4y^2)^4 - 12x}{14y + 40y(3x^2+4y^2)^4}$$

2. Finding the t that corresponds to $P(1,2)$:

$$1 = 5t^2 - 5t + 1 \qquad\qquad 2 = 2t^3 - 4t^2 + t + 2.$$

From the first simpler equation

$$0 = 5t^2 - 5t \quad \text{so} \quad t = 0 \text{ or } t = 1.$$

Checking the y value:

$$2 \neq 2 - 4 + 1 + 2 \qquad\qquad 2 = 2\cdot 0 - 4\cdot 0 + 0 + 2$$

So, $t = 0$. The slope of the tangent is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2 - 8t + 1}{10t - 5}$$

the slope at P is

$$\left.\frac{6t^2 - 8t + 1}{10t - 5}\right|_{t=0} = -\frac{1}{5}$$

Equation of ℓ_1 is

$$y - 2 = -\frac{1}{5}(x - 1)$$

To find the Q solve

$$2t^3 - 4t^2 + t + 2 - 2 = -\frac{1}{5}(5t^2 - 5t + 1 - 1)$$

which simplifies to

$$10t^3 - 15t^2 = 5t^2(2t - 3) = 0$$

So at Q we have $t = 3/2$. It has coordinates

$$5(3/2)^2 - 5(3/2) + 1 = \frac{19}{4} \qquad\qquad 2(3/2)^3 - 4(3/2)^2 + (3/2) + 2 = \frac{5}{4}.$$

The slope is

$$\left.\frac{6t^2 - 8t + 1}{10t - 5}\right|_{t=3/2} = \frac{1}{4}$$

So the equation of ℓ_2 is

$$y - \frac{5}{4} = \frac{1}{4}(x - \frac{19}{4})$$

3. We need to tangent line to $f(x) = \sqrt{x}$ at $x = 16$ for the approximation:

$$f(x) = \sqrt{x} \qquad f(16) = \sqrt{16} = 4$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad f'(16) = \frac{1}{2\sqrt{16}}$$

So the tangent line is

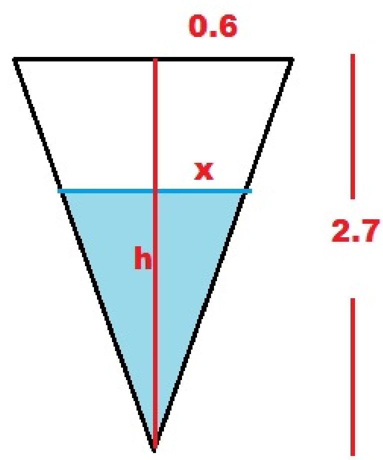
$$y - 4 = \frac{1}{8}(x - 16)$$

and the approximation is

$$\sqrt{15} - 4 \approx \frac{1}{8}(15 - 16) = -0.125$$

so $\sqrt{15} \approx 3.875$.

4. On the right is the cross-section of the tank. Using similar triangles: $\frac{0.6}{x} = \frac{2.7}{h}$ so $x = \frac{2h}{9}$.
The volume of water in the tank is



$$V = \frac{1}{2}2xh(5) = 5h\left(\frac{2h}{9}\right) = \frac{10h^2}{9}$$

Differentiate to get

$$\frac{dV}{dt} = \frac{20h}{9}\frac{dh}{dt}$$

When $V = 2.5 = \frac{10h^2}{9}$, $h = 1.5$ so

$$4.5 = \frac{20(1.5)}{9}\frac{dh}{dt}$$

so $\frac{dh}{dt} = \frac{27}{20}$ meters per minute.