Solutions to Math 124 G Fall 2023 Midterm I

1. (a) $f'(x) = 12x^3 + \dfrac{\pi}{x^2} + 4\pi x^{4\pi-1} - \dfrac{35}{2}\sqrt{x^3}$

   (b) $g'(x) = \left(9x^2 - 16x^3\right)(5\tan x - 6e^x + 9) + \left(3x^3 - 4x^4\right)\left(5\sec^2 x - 6e^x\right)$

   (c) $h'(x) = \dfrac{(9x^2-7)(x\cos x) - (3x^3 - 7x + 1)(\cos x - x\sin x)}{(x\cos x)^2}$

2. (a) $\displaystyle\lim_{t\to 0} \frac{2\cos t - \sqrt{4\cos^2 t + 13\sin^2 t}}{23\sin^2 t} = \lim_{t\to 0} \frac{2\cos t - \sqrt{4\cos^2 t + 13\sin^2 t}}{23\sin^2 t} \cdot \frac{2\cos t + \sqrt{4\cos^2 t + 13\sin^2 t}}{2\cos t + \sqrt{4\cos^2 t + 13\sin^2 t}}$

   $\displaystyle = \lim_{t\to 0} \frac{4\cos^2 t - 4\cos^2 t - 13\sin^2 t}{23\sin^2 t\left(2\cos t + \sqrt{4\cos^2 t + 13\sin^2 t}\right)} = \lim_{t\to 0} \frac{-13}{23\left(2\cos t + \sqrt{4\cos^2 t + 13\sin^2 t}\right)} = -\frac{13}{92}$

   (b) $\displaystyle\lim_{t\to 7} \frac{t-7}{t^2-49} = \lim_{t\to 7} \frac{t-7}{(t-7)(t+7)} = \lim_{t\to 7}\frac{1}{t+7} = \frac{1}{14}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{5x^2+1}}{7x+8} = \lim_{x\to\infty} \frac{\frac{\sqrt{5x^2+1}}{x}}{\frac{7x+8}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{5x^2+1}{x^2}}}{\frac{7x}{x} + \frac{8}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{5x^2}{x^2} + \frac{1}{x^2}}}{7 + \frac{8}{x}} = \lim_{x\to\infty} \frac{\sqrt{5 + \frac{1}{x^2}}}{7 + \frac{8}{x}} = \frac{\sqrt{5}}{7}$

3. (a) We have $f(1) = -2$, $f'(x) = 3x^2 - 4x + 3$, and $f'(1) = 2$ so the tangent to f has equation

   $$y + 2 = 2(x-1).$$

   The tangent to g has the same point, but slope $-1/2$ so its equation is

   $$y + 2 = -\frac{1}{2}(x-1).$$

   (b) From

   $$-2 = g(1) = \frac{a}{b+1}$$

   we get $a = -2(b+1)$. From

   $$g'(x) = -\frac{a}{(b+x)^2}$$

   and

   $$-\frac{1}{2} = g'(1) = -\frac{a}{(b+1)^2}$$

   we get $a = \frac{1}{2}(b+1)^2$. Solving, we get $b = -1$ or $b = -5/2$. We see that $b = -1$ makes g undefined at $x = 1$ which is not the case. So $b = -5$, $a = 8$ and

   $$g(x) = \frac{8}{-5+x}.$$

   (c) The graph of $y = f(x)$ is given by **1** and the graph of $y = g(x)$ is given by **4**. The tangent line to $y = f(x)$ is **2** and has equation $y = 2x - 4$. The tangent line to $y = g(x)$ is **3** and has equation $y = -\frac{1}{2}x - \frac{3}{2}$.

4. (a) $x = -2, 7$

   (b) $x = 0, 4$

   (c) $(-\infty, -2)$, $(-2, 0)$, $(4, 7)$

   (d) $(2, 7)$