Solutions to Math 124 F Winter 2023 Midterm I

1. (a) $\lim_{x\to 2} \frac{x^3-4x}{x^2+x-6} = \lim_{x\to 2} \frac{x(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x(x+2)}{x+3} = \frac{8}{5}$.

   (b) $\lim_{t\to \frac{\pi}{2}} \frac{\sqrt{\sin^2 t + a\cos^2 t} - \sin t}{\cos^2 t} \cdot \frac{\sqrt{\sin^2 t + a\cos^2 t} + \sin t}{\sqrt{\sin^2 t + a\cos^2 t} + \sin t} = \lim_{t\to \frac{\pi}{2}} \frac{\sin^2 t + a\cos^2 t - \sin^2 t}{\cos^2 t\left(\sqrt{\sin^2 t + a\cos^2 t} + \sin t\right)}$

   $= \lim_{t\to \frac{\pi}{2}} \frac{a}{\sqrt{\sin^2 t + a\cos^2 t} + \sin t} = \frac{a}{2}$.

   (c) No. Horizontal asymptotes are determined by computing the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ so there cannot be more than two.

2. (a) $f(x) = \frac{2}{3}e^x + 2x^{-1/2} - \frac{5}{6}x^{-2} + 7x^e$ so $f'(x) = \frac{2}{3}e^x - x^{-3/2} + \frac{5}{3}x^{-3} + 7ex^{e-1}$

   (b) From $g'(x) = \frac{(6x^2 + 4\sec^2 x)(5x^6 + 7\cos x) - (2x^3 + 4\tan x)(30x^5 - 7\sin x)}{(5x^6 + 7\cos x)^2}$ we have $g'(0) = \frac{4}{7}$. Since $g(0) = 0$ the tangent line is $y = \frac{4}{7}x$.

   (c) $\lim_{x\to a} \frac{\tan x - \tan a}{x-a} = \frac{d}{dx}\tan x\Big|_{x=a} = \sec^2 a$.

3. First,

$$\frac{x^3-27}{x-3} = \frac{(x-3)(x^2+3x+9)}{x-3} = x^2+3x+9, \qquad \text{when} \qquad x < 3.$$

So, continuity at $x = 3$ gives

$$3^2 + 3\cdot 3 + 9 = \frac{3b+1}{2\cdot 3 - 1} + a$$

Then,

$$f'(x) = \begin{cases} 2x+3, & x < 3 \\ \dfrac{-b-2}{(2x-1)^2}, & x > 3. \end{cases}$$

So, differentiability at $x = 3$ gives

$$2\cdot 3 + 3 = \frac{-b-2}{(2\cdot 3 - 1)^2}.$$

Solving these equations for a and b we get $a = 163$ and $b = -227$.

4. If $(x, x^3 - x + 1)$ is the point of tangency, writing the slope in two ways, as the slope of the tangent line using the derivative and as rise over run we get the equation

$$3x^2 - 1 = \frac{x^3 - x + 1 - 3}{x + 2}$$

which simplifies to

$$2x^3 + 6x^2 = 0$$

giving us $x = 0$ or $x = -3$ with slopes $m = -1$ and $m = 26$, respectively. So the equations of the tangent lines are

$$y - 3 = -(x + 2) \quad \text{and} \quad y - 3 = 26(x + 2).$$