Math 124 C Fall 2022 Solutions to Midterm II

1. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following.

   (a) $y = \sin\left(x^3 + e^{4x}\right) + \ln(x^2+1)$

$$\frac{dy}{dx} = \cos\left(x^3 + e^{4x}\right) \cdot (3x^2 + 4e^{4x}) + \frac{2x}{x^2+1}$$

$$\frac{d^2y}{dx^2} = -\sin\left(x^3 + e^{4x}\right) \cdot (3x^2 + 4e^{4x})^2 + \cos\left(x^3 + e^{4x}\right) \cdot (6x + 16e^{4x}) + \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

   (b) $x = 3t^2 + e^t$ and $y = 5t + \cos t$

$$\frac{dy}{dx} = \frac{5 - \sin t}{6t + e^t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{5-\sin t}{6t+e^t}\right)}{\frac{dx}{dt}} = \frac{\frac{-\cos t(6t+e^t) - (5-\sin t)(6+e^t)}{(6t+e^t)^2}}{6t+e^t} = \frac{-\cos t(6t+e^t) - (5-\sin t)(6+e^t)}{(6t+e^t)^3}$$

2. Find the equation of the tangent line to $y = f(x)$ at $x = 3$ for the following.

   (a) $y = \sqrt{2x + \sqrt{x+6}}$

$$y' = \frac{1}{2\sqrt{2x + \sqrt{x+6}}}\left(2 + \frac{1}{2\sqrt{x+6}}\right)$$

$$y'(3) = \frac{1}{2\sqrt{6+\sqrt{9}}}\left(2 + \frac{1}{2\sqrt{9}}\right) = \frac{1}{6}\left(2 + \frac{1}{6}\right) = \frac{13}{36}$$

$$y(3) = \sqrt{6 + \sqrt{9}} = \sqrt{9} = 3$$

   Tangent line:

$$y - 3 = \frac{13}{36}(x-3)$$

   (b) $y = (x-2)^{x^2-8}$

$$\ln y = \ln\left((x-2)^{x^2-8}\right) = (x^2-8)\ln(x-2)$$

$$\frac{y'}{y} = 2x\ln(x-2) + \frac{x^2-8}{x-2}$$

$$y' = \left(2x\ln(x-2) + \frac{x^2-8}{x-2}\right)(x-2)^{x^2-8}$$

$$y(3) = (3-2)^{9-8} = 1$$

$$y'(3) = \left(6\ln(1) + \frac{1}{1}\right)(3-2)^{9-8} = 1$$

   Tangent line

$$y - 1 = 1 \cdot (x-3)$$

   or $y = x - 2$.

3. (a) Implicit differentiation for $4xy^2 - 4y + 5x^2 = 24$

$$4y^2 + 4x \cdot 2yy' - 4y' + 10x = 0$$

When $x = 2$, $y = 1$,

$$4 + 16y' - 4y' + 20 = 0$$

so $y' = -2$. The tangent line is

$$y - 1 = -2(x - 2)$$

or $y = -2x + 5$.

(b) Intersecting the line with the curve:

$$4x(-2x+5)^2 - 4(-2x+5) + 5x^2 = 24$$

$$16x^3 - 75x^2 + 108x - 44 = 0$$

We know $x = 2$ is a solution so

$$(x-2)(16x^2 - 43x + 22) = 0$$

Factoring the quadratic:

$$(x-2)(-2)(16x-11) = 0$$

we get $x = 11/16$.

(c) The approximation is $y \approx -2x + 5$ so

$$y \approx -2 \cdot \frac{11}{16} + 5 = \frac{29}{8} = 3.625$$

4. Let h be the depth of the water and r be the radius at the surface of the water. We know $\frac{dV}{dt} = 2$ and we want $\frac{dh}{dt}$. The volume of water is

$$V = \frac{1}{3}\pi r^2 h$$

from similar triangles

$$\frac{3}{r} = \frac{7}{h}$$

so $r = \frac{3h}{7}$ giving the volume of water

$$V = \frac{1}{3}\pi\left(\frac{3h}{7}\right)^2 h = \frac{3\pi}{49}h^3$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{9\pi}{49}h^2\frac{dh}{dt}$$

When the tank is full

$$V = \frac{1}{3}\pi 3^2 \cdot 7 = 21\pi$$

so when the tank is half full

$$\frac{21\pi}{2} = \frac{3\pi}{49}h^3$$

so

$$h = \frac{7}{\sqrt[3]{2}}.$$

Then,

$$2 = \frac{9\pi}{49}\left(\frac{7}{\sqrt[3]{2}}\right)^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{2^{5/3}}{9\pi}$$