Problems for the UW Putnam Session 7, November 23, 2009

Play with a few of the problems below. If some are too easy (and you know how to solve them), move on – you’re bound to hit some that will challenge you. These problems are mostly about sequences, series, derivatives and integrals.

If you don’t know how to approach a problem, try a few small cases. Look for patterns. Draw a picture. Work Backward. Divide into cases. Don’t give up after 2 minutes.

This is left-over from 11/09.

Problem 1.1. Let \( n \geq 2 \) be a fixed integer. Find the smallest constant \( C \) such that, for any non-negative real numbers \( x_1, x_2, \ldots, x_n \),

\[
\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left( \sum_{i=1}^{n} x_i \right)^4.
\]

For that value of \( C \), what numbers \( x_1, \ldots, x_n \) achieve the equality?

Sequences and limits: Left-overs from last time.

Problem 1.2. Define \( a_0 = a \), \( a_{n+1} = 2a_n - n^2 \) for all \( n \geq 0 \). For what values \( a \) is \( a_n \geq 0 \) for all \( n \)?

Problem 1.3. Find

\[
\lim_{n \to \infty} \left[ \frac{1}{n} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n} \right].
\]

Problem 1.4. Define the triangular sequence \( a_{i,0} = \frac{x}{2} \) for all \( i \geq 0 \), and \( a_{i,j+1} = a_{i,j}^2 + 2a_{i,j} \) for all \( i, j \geq 0 \). What is \( \lim_{n \to \infty} a_{n,n} \)?

Problem 1.5. Let \( b_n \in \{-1, 1\} \) for all \( n \geq 1 \), and \( b_n = b_{8+n} \) for all \( i \geq 1 \). Let \( a_n = \frac{b_n}{n} \). If exactly 4 of the values \( b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \) are positive, show that the series \( S = \sum_{n=1}^{\infty} a_n \) converges. Is the reverse true?

Problem 1.6. If \( 0 < a < b \), calculate

\[
\lim_{t \to 0} \left[ \int_{0}^{1} (bx + a(1-x))^t \, dx \right]^{1/t}.
\]
Problem 1.7. Show that, if we independently choose two positive integers at random, the probability that they are co-prime is \( \frac{6}{\pi^2} \).

New problems: probability, functional relations, etc.

Problem 1.8. Let \( k > 1 \). Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = f(x^2 + k) \) for all \( x \).

Problem 1.9. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( f(2x^2 - 1) = 2xf(x) \). Show \( f(x) = 0 \) for all \( x \in [-1, 1] \).

Problem 1.10. What is the probability that \( 2^n \) starts with 1 in its decimal expression?

Problem 1.11. Given an irrational number \( p \in (0, 1) \), is there a coin-flipping game that, with probability 1, ends in a finite number of steps, and wins with probability \( p \)?

Problem 1.12. Let \( f : \mathbb{N} \to \mathbb{N} \) a function such that \( f(n + 1) > f(f(n)) \), for all \( n \). Prove that \( f(n) = n \).

Problem 1.13. Compute:
\[
\lim_{x \to 1} \prod_{n=0}^{\infty} \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.
\]

Problem 1.14. Find the probability for a randomly chosen positive integer \( n \) to have a prime divisor greater than \( \sqrt{n} \).