Problems for the UW Putnam Session, Session 4

November 2, 2009

Play with a few of the problems below. If some are too easy (and you know how to solve them), move on – you’re bound to hit some that will challenge you. These problems are mostly about Number Theory, Geometry, and Trigonometry.

If you don’t know how to approach a problem, try a few small cases. Look for patterns. Draw a picture. Work Backward. Divide into cases. Don’t give up after 2 minutes.

1 Geometry

These are left-overs from last Monday.

Problem 1.1. Consider the function $y = x^2$ on the interval $[0,1]$. Let $t \in [0,1]$, and consider the line $y = t$. Let $S_1$ be the area enclosed between the $y$-axis, the parabola and the line $y = t$, and $S_2$ be the area enclosed between the vertical line $x = 1$, the parabola, and the line $y = t$. For which values of $t$, $S_1 + S_2$ is (a) minimal, (b) maximal?

Problem 1.2. A circle of radius $r$ is rolling inside the circle of Radius $R > r$. Fix a point $M$ on the smaller circle. The trajectory of this point is called a hypercycloid. Show that for $r = 1/2R$, the hypercycloid degenerates to an interval.

Problem 1.3. Find the volume of the region of points $(x,y,z)$ such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Problem 1.4. What is the maximal length of a pole that can be moved from a hallway of width $a$ to a hallway of width $b$ horizontally (that is, the pole must stay parallel to the ground. Assume it has zero width)?

Problem 1.5. Right triangle $ABC$ has a right angle at $C$ and $\angle BAC = \theta$; the point $D$ is chosen on $AB$ such that $AC = AD = 1$; the point $E$ is chosen on $BC$ so that $\angle CDE = \theta$. The perpendicular to $BC$ at $E$ meets $AB$ at $F$. Evaluate $\lim_{\theta \to 0} |EF|$.

Problem 1.6. Prove that the interval $[-1,1]$ cannot be expressed as a union of two disjoint congruent subsets (equal means that they differ by a translation).

And new problems:
Problem 1.7. Let $ABC$ be a triangle whose vertices lie on the sides of a parallelogram $EDFG$. Prove that the area of the triangle $ABC$ is no greater than half of the area of the parallelogram.

Problem 1.8. Let $n$ be a positive integer, $n \geq 2$, and let $\theta = 2\pi/n$. Define points $P_k = (k,0)$, for $k = 0,1,\ldots,n$. Let $R_k$ be the rotation of the plane counterclockwise by the angle $\theta$ about the point $P_k$. For an arbitrary point $(x,y)$ find a simple formula for the coordinates of the composition $R_n \circ R_{n-1} \circ \ldots \circ R_2 \circ R_1(x,y)$.

2 Number theory

Definition 2.1. Let $n$ be a positive integer. Define the following number-theoretic functions:

- $d(n)$ = the number of divisors of $n$ which are smaller or equal to $n$;
- $\sigma(n)$ = the sum of all divisors of $n$ which are smaller or equal to $n$;
- $\phi(n)$, the Euler function, is the number of integers between 1 and $n$ which are coprime to $n$.

Example 2.1. $d(6) = 4$, $\sigma(6) = 12$, and $\phi(6) = 2$.

Definition 2.2. A number theoretic function $f : \mathbb{N} \to \mathbb{N}$ is called multiplicative if $f(ab) = f(a)f(b)$ for $(a,b) = 1$.

The functions $d(n), \sigma(n)$ and $\phi(n)$ are multiplicative. This is a useful thing to know to prove their formulas. Let $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$.

Then

$$d(n) = (k_1 + 1)(k_2 + 1)\ldots(k_m + 1)$$

$$\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \ldots \frac{p_m^{k_m+1} - 1}{p_m - 1}$$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_m}\right)$$

We have three problems left from last Monday:

Problem 2.1. Let $n > 1$. Show that $S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is not an integer.

Problem 2.2. Show that $n$ does not divide $2^n - 1$ for any $n > 1$.

Problem 2.3. Show that $n^2/2 < \sigma(n)\phi(n) < n^2$.

And some new problems:

Problem 2.4. Prove that from the set $\{0,1,2,\ldots,3^k-1\}$ one can choose $2^k$ numbers so that none of them can be represented as the arithmetic mean of some pair of the chosen numbers.

Problem 2.5. Prove that $a_{m,n} = \frac{(2m)! (2n)!}{m!n!(m+n)!}$ is an integer for any two non-negative integers $m,n$.

Problem 2.6. Let $f(n) = 1 + 2n + 3n^2 + \ldots + (p-1)n^{p-2}$, where $p$ is an odd prime. Prove that if $f(m) \equiv f(n) \pmod{p}$, then $m \equiv n \pmod{p}$.