11 Problems for the UW Putnam Session. Issued 8 Nov. 2010

Play with a few of the problems below. If some are too easy (and you know how to solve them), move on – you’re bound to hit some that will challenge you. These problems are mostly about Inequalities (see handout).

If you don’t know how to approach a problem, try a few small cases. Look for patterns. Draw a picture. Work Backward. Divide into cases. Don’t give up after 2 minutes.

Problems Left Over From Previous Sessions

Problem 1. Find the maximal possible value of $GCD(ab + 1, ac + 1, bc + 1)$ if $a, b, c$ are distinct natural numbers such that $a + b + c \leq 3000000$.

Problem 2. The triangle ABC has a right angle at C and the angle $\angle BAC = \theta$ (see Figure). The point D is chosen on AB such that $AD = AC = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Find $\lim_{\theta \to 0} EF$.

Problem 3. Show that there exists no non-constant polynomial $P(x)$ with integer coefficients such that $P(n)$ is prime for all positive integers $n$.

Problem 4. Show that if $n$ is an integer greater than 1, then $n$ does not divide $2^n - 1$.

New Problems.

Problem 5. Let $a$ be a positive constant. Find the minimum value for the function $f(x) = \frac{a^2 + x^2}{x}$, defined for $x \in (0, \infty)$.

Problem 6. Let $a, b, c, \alpha, \beta, \gamma$ be real numbers such that $a^2 + b^2 + c^2 = \alpha^2 + \beta^2 + \gamma^2 = 1$. Show that $-1 \leq \alpha a + \beta b + \gamma c \leq 1$.

Problem 7. Suppose $a, b, c$ are positive real numbers. Prove that

$$\sqrt[3]{abc} + 1 \leq \sqrt[3]{(a + 1)(b + 1)(c + 1)} \leq \left(\frac{\sqrt[3]{a+1} + \sqrt[3]{b+1} + \sqrt[3]{c+1}}{3}\right)^3 \leq \frac{a + b + c}{3} + 1.$$
Problem 8. Let $m, n$ be positive integers. Show that
\[
\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^mn^n}.
\]

Problem 9. Suppose there are $N$ students in a class. On Day 1, they are divided into $n$ groups to work on problems. On Day 2, they are divided into $n + 1$ groups, for the same purpose. Show that there exist at least two students who, on Day 2, were in groups of smaller size than the groups they had been in on Day 1.

Problem 10. Let $x_1, x_2, \ldots, x_n$ be a permutation of $1, 2, \ldots, n$. Find, with proof, the largest value that $x_1x_2 + x_2x_3 + x_3x_4 + \ldots + x_{n-1}x_n + x_nx_1$ can take, as a function of $n$ ($n \geq 2$).

Problem 11. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative numbers. Show that
\[
(a_1a_2\ldots a_n)^{1/n} + (b_1b_2\ldots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2)\ldots(a_n + b_n))^{1/n}.
\]