Let $G$ be a bipartite graph with two equal-sized classes $X$ and $Y$ $(|X| = |Y| = n)$. Assume that all vertices of $G$ have degree at least $n/2$. Show the following.

a) For any set $S \subset X$ of size $|S| > \frac{n}{2}$, $N_G(S) = Y$.

b) Show that $G$ has a perfect matching.

Let $m, n$ be positive integers and let $A_1, \ldots, A_m$ be a partition of $[mn] = \{1, 2, \ldots, mn\}$ such that $|A_i| = n$ for all $i$. Let now $B_1, \ldots, B_m$ be a second partition of $[mn]$ such that $|B_i| = n$ for all $i$. Show that there exists a permutation $\sigma$ of $[n]$ such that $A_i \cap B_{\sigma(i)} \neq \emptyset$ for all $i$.

Recall that a Euler closed trail is a trail which uses each edge of the graph exactly once.

a) Let

$$A_G = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}$$

be the adjacency matrix for the graph $G$. Does $G$ have an Euler closed trail?

b) Now let

$$A_{G'} = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 2 \\
1 & 0 & 0 & 1 & 2 & 0 \\
\end{bmatrix}$$

be the adjacency matrix for the graph $G'$. Does $G$ have an Euler closed trail?

Let $T$ be a finite tree in which each vertex has degree either 1 or 3. Prove that

a) the number of edges is odd, and

b) more than half of the vertices are leaves.

Let $G$ be an undirected connected graph with at least 2 vertices. Prove that there is some vertex $v$ of $G$ such that we can remove it, along with all of the edges incident to it (coming into it), and the resulting graph is connected.
P6 Let $K_n$ be the complete graph on $n$ vertices. How many of its spanning trees have $n$ as a leaf? (Note: you will need Cayley’s theorem.)

P7 How many spanning trees does the graph in the picture have? (Hint: when you calculate the Laplacian matrix, choose the vertex you decide to leave out wisely.)

P8* How many spanning trees does the graph in the picture have?

P9 Give an example, or state why one cannot be found. Read carefully and think before you answer.

a) A disconnected graph with 10 vertices and chromatic number 9.

b) A graph with 7 vertices that is not $K_7$, but has chromatic number 7.

c) (for arbitrary $n \geq 6$) A graph with $n$ vertices and the degree of each vertex less than or equal to 2, and with chromatic number 3.
d) A planar graph with 10 vertices, 10 edges, 3 faces.

\textit{P10}^* Calculate the chromatic number of \( C_4 \); calculate the chromatic number of the wheel \( W_n \).

\textbf{P11}

a) Is the bipartite graph \( K_{2,n} \) planar? If so, draw a planar representation of the graph.

b) Do the same for the complete tripartite graph \( K_{2,2,2} \).

\textbf{P12} Let \( Q_n \) be the \( n \)-dimensional hypercube, i.e., the graph with vertex set

\[ V(Q_n) = \{ x : x \text{ is a length } n \text{ string of 0s and 1s} \} . \]

For example, \( Q_2 \) has vertices 00, 01, 10, and 11. We connect two vertices if they differ in \textit{exactly} 1 position; e.g., for \( Q_2 \), we connect 00 to 01 and 10, 01 connects to 00 and 11, etc. and we get that \( Q_2 \) is the square (similarly, \( Q_3 \) is the cube).

1. What is the chromatic number of \( Q_n \)?

2. What is the chromatic polynomial of \( Q_2 \)?

\textbf{P13} (essentially Problem 26, chapter 11) Let \( S = [n] = \{1, 2, \ldots, n\} \). Let \( k \) be an integer so that \( \frac{n-1}{2} \geq k \). Let \( X_k \) and \( Y_k \) be the sets of all subsets of \( S \) with \( k \) elements, respectively, with \( k + 1 \) elements. Construct a bipartite graph with \( X \) and \( Y \) as classes, and draw an edge between vertices \( x \in X_k \) and \( y \in Y_k \) if \( x \subset y \).

Show that there exists a perfect matching from \( X \) into \( Y \).

\textbf{P14}

a) Mark the minimum spanning tree on Figure 1 (next page), and calculate its total weight.

b) Suppose that a weighted graph \( G \) has 10 minimum spanning trees. Let \( T \) be one of them, and let \( (u, v) \) be an edge in \( T \). Suppose we construct the graph \( G' \), which can be obtained from \( G \) by removal of the edge \( (u, v) \). How many spanning trees is \( G' \) guaranteed to have?
Figure 1: $AC = 10$, $AE = 6$, $BC = 8$, $BD = 16$, $BF = 13$, $CD = 7$, $CE = 17$, $CF = 11$, $DE = 5$, $EF = 14$