In the below, \( F \in \{\mathbb{R}, \mathbb{C}\} \).

**P1 (20pts).**

a) Find the best constants \( m, M \) so that
\[
m|||A|||_1 \leq |||A|||_2 \leq M|||A|||_1 , \quad \text{for } A \in \mathcal{M}_n .
\]
b) Solve the above, this time for the Frobenius norm \( |||A|||_F \).

**P2 (20pts).**

a) If \( A \in \mathcal{M}_n \) is singular, explain why \( |||I_n - A||| \geq 1 \) for any choice of matrix norm \( ||| \cdot ||| \).

b) Give an example of a matrix \( A \) such that \( \rho(A) < |||A||| \) for any matrix norm \( A \).

c) Prove that, for any matrix \( A \in \mathcal{M}_n ,
\[
|||A|||_2^2 \leq |||A|||_1 |||A|||_\infty .
\]
d) Explain why the function \( ||| \cdot ||| \) defined on \( \mathcal{M}_n \) by \( |||A||| = n \max_{i,j} |a_{ij}| \) is a matrix norm.

**P3 (30pts).** Let \( A \in \mathcal{M}_n \), nonsingular. We define the concept of the condition number of a matrix \( A \) with respect to any matrix norm as \( \kappa_{||| \cdot |||}(A) = |||A||| \cdot |||A^{-1}||| \).

a) Let \( |\lambda_{\max}| = \rho(A) \) and \( |\lambda_{\min}| \) be the absolute value of the eigenvalue of \( A \) closest to 0. Show that for any norm \( ||| \cdot ||| \), \( \kappa_{||| \cdot |||}(A) \geq \frac{|\lambda_{\max}|}{|\lambda_{\min}|} \).

b) In addition, assume that \( B \in \mathcal{M}_n \) is singular. Show that for any norm \( ||| \cdot ||| \), \( \kappa_{||| \cdot |||}(A) \geq \frac{|||A|||}{|||A-B|||} \).

c) Use b) to show that if \( A \) is invertible and upper triangular, then the condition number of \( A \) with respect to \( ||| \cdot |||_\infty \) satisfies
\[
\kappa_{||| \cdot |||_\infty}(A) \geq \frac{|||A|||_\infty}{\min_{1 \leq i \leq n} |a_{ii}|} .
\]

**P4 (30pts).** The following problem involves the companion matrix \( M_p \) for the monic polynomial \( p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \).
a) Use the norm $||| \cdot |||_\infty$ to deduce Cauchy’s bound: any root $z$ of $p$ lies in the disk with radius $1 + \max_i |a_i|$ and center at the origin.

b) Use the $||| \cdot |||_1$ norm to prove Montel’s bound: any root $z$ of $p$ lies in the disk of radius $\max\{1, \sum_{i=0}^{n-1} |a_{ii}|\}$ and center at the origin.

c) Obtain the following, different bound of Montel’s by using the polynomial $q(z) = (z - 1)p(z)$ to obtain that any root $z$ of $p$ lies in the disk centered at the origin and of radius $|a_0| + \sum_{i=1}^{n-1} |a_{i-1} - a_i| + |a_{n-1} - 1|$. 