In the below, $F \in \{\mathbb{R}, \mathbb{C}\}$.

**P1 (10pts).** Suppose that the matrices $A, B \in M_n(F)$ commute. Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be the eigenvalues of $A$, respectively, $B$ (with multiplicities). Show that there exist orderings $\sigma$ and $\pi$ of these two sets of eigenvalues so that the eigenvalues of $A + B$ and $AB$ are, respectively, $\lambda_{\sigma(i)} + \mu_{\pi(i)}$ and $\lambda_{\sigma(i)}\mu_{\pi(i)}$, for $i = 1, \ldots, n$.

**P2 (30pts).**

a) Prove that a matrix $A$ is nilpotent iff it is similar to a strictly upper triangular matrix.

b) If $A$ and $B$ are nilpotent, must $A + B$ be nilpotent? What about $AB$? Prove your assertions by argument or by example.

c) Suppose $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Then $A$ is nilpotent iff $A_{11}$ and $A_{22}$ are nilpotents.

**P3 (15pts).** Let $A$ be a matrix of rank $r$. Show that there exists a polynomial $p$ of degree $r + 1$ such that $p(A) = 0$. You may use only theorems we proved in class (i.e., you may not use the Jordan canonical form or minimal polynomials).

**P4 (10pts).** Show that two matrices $A$ and $B$ have the same eigenvalues (with the same multiplicities) if and only if $\text{tr}(A^k) = \text{tr}(B^k)$ for all $1 \leq k \leq n$. Conclude that $A$ is nilpotent iff $\text{tr}(A^k) = 0$ for all $1 \leq k \leq n$.

**P5 (25pts).** Let $A, B \in M_n$ and suppose that $A$ and $B$ are simultaneously triangularizable, with $A = ST_A S^{-1}$ and $B = ST_B S^{-1}$, with the diagonal of $T_A$ being $(\alpha_1, \ldots, \alpha_n)$ and the diagonal of $T_B$ being $(\beta_1, \ldots, \beta_n)$.

a) Prove that the polynomial $p_{A,B}(s,t) := \det(tB - sA)$ is $p_{A,B}(s,t) = \prod_{i=1}^n(t\beta_i - s\alpha_i)$.

b) Deduce that

$$p_{A,B}(B,A) := \prod_{i=1}^n(\beta_i A - \alpha_i B) = S \prod_{i=1}^n(\beta_i T_A - \alpha_i T_B) S^{-1}.$$

c) Prove the generalized Cayley-Hamilton, i.e. $p_{A,B}(B,A) = 0$ (Hint: use the multiplication lemma we used to prove Cayley-Hamilton).

**Remark 1.** For Problem 5, in H-J, you will find a commutativity requirement for $A, B$ in order to prove parts b) and c). I believe they would have liked you to start from the definition of $p_{A,B}(s,t)$ as a polynomial in two commuting variables, and work from there. But if you start from the definitions I provided, you should not need commutativity.