**P1** Let $A_n$ be the graph obtained from $K_n$ by deleting an edge. How many spanning trees does $A_n$ have?

Suppose wlog that the missing edge is between $n$ and $n-1$. Set up the Laplacian-like matrix to ignore $n$, it looks like:

$$L = \begin{bmatrix}
  n-1 & -1 & -1 & \cdots & -1 \\
  -1 & n-1 & -1 & \cdots & -1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & -1 & \cdots & n-2
\end{bmatrix},$$

since $n-1$ has degree $n-2$, not $n-1$ like the other vertices. To compute the determinant, first add rows 2 through $(n-1)$ to row 1 to get:

$$\det(L) = \begin{vmatrix}
  1 & 1 & 1 & \cdots & 0 \\
  -1 & n-1 & -1 & \cdots & -1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & -1 & \cdots & n-2
\end{vmatrix},$$

and then add row 1 to rows 2 through $(n-1)$ to get:

$$\det(L) = \begin{vmatrix}
  1 & 1 & 1 & \cdots & 0 \\
  0 & n & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & n-2
\end{vmatrix};$$

as this matrix is upper triangular, the determinant is the product of its diagonal values, and that makes it $1 \cdot n^{n-3} \cdot (n-2)$. That’s the number of spanning trees of $A_n$.

**P2** How many spanning trees does the complete bipartite graph $K_{m,n}$ have?

Label the left set of vertices 1 through $m$, and the right one $m+1$ through $m+n$. Set up the Laplacian matrix, ignoring the very last vertex, $m+n$. The result looks like this:

$$\det(L) = \begin{vmatrix}
  n & 0 & \cdots & 0 \\
  0 & n & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & n \\
  -1 & -1 & \cdots & -1 \\
  -1 & -1 & \cdots & -1 \\
  \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & \cdots & -1 \\
  m & 0 & \cdots & 0 \\
  0 & m & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & m
\end{vmatrix},$$
with the note that, since we deleted the last row and column, the multiple of the identity in
the (2, 2) position above is only \((n - 1) \times (n - 1)\).

Adding rows 2 through \(n + m - 1\) to the first row, we obtain

\[
\det(L) = \begin{vmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & n & \ldots & 0 & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & n & -1 & -1 & \ldots & -1 \\
-1 & -1 & \ldots & -1 & m & 0 & \ldots & 0 \\
-1 & -1 & \ldots & -1 & 0 & m & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \ldots & -1 & 0 & 0 & \ldots & m \\
\end{vmatrix},
\]

since for \(1 \leq i \leq m\), rows \(m + 1\) through \(m + n - 1\) each contain a \(-1\) in position \(i\) and row \(i\) contains an \(n\), and all other rows have \(i\) entries 0; respectively, for \(m + 1 \leq i \leq m + n - 1\) rows \(1\) through \(m\) contain a \(-1\) in position \(i\), and the \(i\)th row has an \(m\), while all other rows have 0 in position \(i\).

Finally, now add row 1 to all rows from \(m + 1\) through \(m + n - 1\), to get

\[
\det(L) = \begin{vmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & n & \ldots & 0 & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & n & -1 & -1 & \ldots & -1 \\
0 & 0 & \ldots & 0 & m & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & m & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & m \\
\end{vmatrix},
\]

and since this matrix is upper triangular, the determinant is the product of the entries on
the diagonal, so \(n^{m-1}m^{n-1}\).

**P3** Let \(G\) be a bipartite graph in which each vertex has degree exactly \(d\). Show that there
are at least \(d\) different perfect matchings in \(G\).

Note that a bipartite, \(d\)-regular graph \(G = (X, Y)\) must satisfy \(d|X| = d|Y|\) and so \(|X| = |Y|\).
We proceed by induction over \(d\). If \(d = 1\), the bipartite, regular graph must be a perfect
matching itself.

Assume now that we have shown that any bipartite, \((d - 1)\)-regular graph has \(d - 1\)
distinct matchings, and consider a bipartite, \(d\)-regular graph. As we have seen before (in the
homework, when we showed this for a bipartite biregular graph), the Hall Theorem conditions
are satisfied, so the graph will have one matching of \(X\) into \(Y\) (and since \(|X| = |Y|\), that will
be a perfect matching). Delete now all the edges in the matching; the resulting graph \(G'\) is
still bipartite, and now it is \((d - 1)\)-regular. By induction it has \((d - 1)\) distinct matchings.
These are also matchings in $G$, and $G$ has one more, the one we deleted. Hence the induction step is proved, and the proof is completed.

**P4** An $n \times n$ matrix is called a *permutation matrix* if all its entries are equal to 0 or 1, and there is precisely one 1 in each row and in each column. (For example, the identity matrix $I_n$ is a permutation matrix.)

a) Given a permutation matrix $P$, consider the bipartite graph $G$ whose left class $X$ is indexed by the rows and the right class $Y$ is indexed by the columns of $P$, and for which $(x, y)$ with $x \in X$ and $y \in Y$ is an edge if and only if $P_{xy} = 1$.

What kind of bipartite graph is $G$?

b) Show that if $M$ is an $n \times n$ matrix with all entries equal to 0 or 1 and with exactly $m$ 1s in each row and exactly $m$ 1s in each column, then $M$ can be written as a sum of $m$ permutation matrices.

**P5** There are $n$ applicants for some set of $m$ jobs. Assume that for any $2 \leq k \leq n$, every $k$ applicants have applied to at least $k - 2$ jobs. Show that it is possible to match $n - 2$ or more of applicants to jobs they applied for.

Construct a bipartite graph with one class $X$ being the applicants and the other one $Y$ the jobs, and with an edge between an applicant and a job if and only if the applicant applied for the job. The conditions of Hall’s Theorem are almost met, but not quite, since all the above says is that for every $S \subseteq X$, $|S| - 2 \leq |N(S)|$.

Add two fictitious jobs, and make every applicant apply to both. The new class $Y'$ is two larger, and now the graph has the property that for every $S \subseteq X$, $|N(S)| \geq |S|$, since we increased the neighborhood of $S$ by 2. Hall’s Theorem’s conditions are met, and so a perfect matching of $X$ into $Y'$ must exist. Ignore the two poor unfortunate souls who get matched with the fictitious jobs—now you have a matching of $n - 2$ of the other applicants to real jobs in $Y$.

**P6** (Supplementary Exercise 19, chapter 11) A graph is called *color critical* if it has chromatic number $k$, but if we delete any vertex of the graph together with its incident edges, we get a graph of chromatic number $k - 1$. 
a) Give an example of a color-critical graph for $k = 3$. The example should not be $K_3$.

b) Given a non-complete graph example of a color-critical graph for $k = 4$. The example should not be $K_4$.

a) An odd cycle. b) A wheel graph with an odd cycle.

**P7** Let $G$ be a graph on 11 vertices, and $G^c$ be its complement. Show that at least one of $G$ and $G^c$ is not planar.

A planar graph on $V$ vertices with $E$ edges must satisfy $E \leq 3V - 6$. Let $E$ and $E'$ be the numbers of edges in $G$, respectively $G^c$. If both were planar we should have $E \leq 3V - 6$ and $E' \leq 3V - 6$ so $E + E' \leq 27$ as $V = 11$. But then $55 = \binom{11}{2} = E + E' \leq 27 \cdot 2 = 54$, contradiction. So one of $G$, $G^c$ must be non-planar.

**P8**

a) Suppose we delete two edges from $K_6$. Is the resulting graph planar? What about if we delete three edges?

b) A planar graph $G$ has 16 vertices and 40 edges. How many triangular regions are there in the planar drawing of $G$, if all the regions (including the unbounded one) are either triangles or quadrilaterals?

a) If the two edges meet at a vertex, the graph still contains a $K_5$. If the edges are disjoint, the graph still contains a $K_{3,3}$. So in either case it isn’t planar. If the three edges meet at a vertex, the graph still contains a $K_5$. If the three edges are mutually disjoint, the graph is planar, and can be drawn as such.

b) Let $F = t + q$, with $t$ representing the number of triangles, and $q$ the number of quadrilaterals. Then $80 = 2E = 3t + 4q$, and from Euler’s formula $t + q + 16 = \frac{3t + 4q}{2} + 2$ we get $t = 24$, $q = 2$.

**P9** Consider a group of 8 people, each pair of which are either friends or enemies. Show that if some person in the group has at least 6 friends in the group, then either there are 4 people who are mutual friends or 3 people who are mutual enemies.

Consider the graph on 8 vertices, each vertex identified with a person, and suppose we color Red edges between friends and Blue edges between enemies. Let $v$ be the vertex with 6 friends (Red edges). The 6 people at the other end of these 6 edges form a $K_6$, and we know that a $K_6$ has to have a monochromatic triangle (by Ramsey theory). If that triangle is Red, add $v$ to its vertices and get a Red $K_4$ (4 people who are mutual friends), else if the triangle is Blue we have found 3 people who are mutual enemies.
P10 Suppose that \( n \) people attend a party. In any group of 3 guests, there are two who do not like each other. In any group of 7 guests, there are two who do like each other. At the end of the party, each person gives a gift to all the people they like.

Show that at most \( 6n \) gifts have been given at the party.

Construct a complete graph on \( n \) vertices, and connect people with a Blue edge if they like each other and with a Red edge if they do not. Pick a vertex \( v \) and assume it has at least 7 neighbors to whom it is connected with a Blue edge. Then, among these 7 neighbors there must be two, call them \( u \) and \( w \), who like each other and so they are also connected by a Blue edge. But the \( (u, w, v) \) make up a Blue triangle, and that contradicts the fact that there should be no Blue triangles. So no vertex can have more than 6 Blue edges, and hence at most \( 6n \) gifts have been given.

P11 Given the complete graph on \( 2n \) vertices \( K_{2n} \), show that there exists a coloring of the edges on \( K_{2n} \) with \( n \) colors which has fewer than \( 4n/3 \) monochromatic triangles.

Color randomly and uniformly with all \( n \) colors. The probability that triangle \((i, j, k)\) is monochromatic is \( n \cdot \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \). There are \( \binom{2n}{3} \) possible triangles, so in expectation there should be \( \frac{\binom{2n}{3}}{n^3} = \frac{2n(2n-1)(2n-2)}{6n^3} = \frac{(2n-1)(2n-2)}{3n} < \frac{4n}{3} \) monochromatic triangles. Hence some coloring should yield that.

P12 Consider the graph \( K_n \), and pick a uniformly random 2-coloring of its edges. How many monochromatic \( K_4 \)s do we expect \( G \) to have?

There are \( \binom{n}{4} \) possible \( K_4 \)s, 2 colors, and once a color is chosen the probability that all 6 edges of a \( K_4 \) are that color in \( 1/2^6 \). Overall this yields \( \frac{n}{32} \).

P13 We say that a permutation \( \pi \) of \([n]\) transposes \( i \) and \( j \) if \( \pi(i) = j \) and \( \pi(j) = i \). For instance, if \( n = 7 \) and \( \pi = (5, 3, 2, 6, 1, 7, 4) \) then \( \pi \) transposes 1 and 5, and it also transposes 2 and 3.

What is the expected number of transpositions in a random permutation of \([n]\)? Assume all permutations are equally likely.

Let \( P_1, P_2, \ldots, P_{\binom{n}{2}} \) be a list of all transpositions, and let \( X_i \) be the indicator variable for transposition \( i \), meaning it is 1 if the permutation contains it and 0 otherwise. So the number of transpositions that a permutation \( \pi \) contains is \( \sum_{i=1}^{\binom{n}{2}} X_i \).

Pick a uniformly random permutation; the probability that it contains transposition \( i \) is given by the number of permutations that contain it over \( n! \), so it is \( (n-2)!/n! = 1/(n(n-1)) \). Hence \( E[X_i] = 1/(n(n-1)) \), and the expected number of transpositions is \( \binom{n}{2} \frac{1}{n(n-1)} = \frac{1}{2} \).

P14 What is the expected number of leaves in a uniformly random tree with vertex set \([n]\), and what fraction of the vertices are expected to be leaves as \( n \to \infty \)?
We use the same idea as above; in a tree $T$ we define the variables $X_1, \ldots, X_n$ which indicate whether $i$ is a leaf. Hence the expected number of leaves is $nE[X_n]$. Recall a problem from the midterm review: the number of trees with $n$ as a leaf is $(n-1)^{n-2}$, so the probability that a random tree has $n$ as a leaf is $(n-1)^{n-2}/n^{n-2} = (1-\frac{1}{n})^{n-2}$, and the expected number of leaves in a uniformly random tree is $n \left(1 - \frac{1}{n}\right)^{n-2}$. Simple calculus shows that the fraction tends to $1/e$ as $n \to \infty$.

P15

1) Let $P$ be the poset of numbers up to $n$ ordered by divisibility ($x \leq y$ if $x$ divides $y$). What is the size of the longest chain in $P$?

2) Recall $B_n$, the poset of the power set of $[n]$ (the set of all subsets of $[n]$) ordered by inclusion. What is the length of the longest chain in $B_n$?

1) $k = \lfloor \log_2 n \rfloor + 1$, as the longest chain is given by $1, 2, 4, \ldots, 2^k$ for which $2^k \leq n < 2^{k+1}$.

2) $n + 1$, as starting from $\emptyset$ we must add at least one new element each time we go up the chain.

P16 Let $I_1, \ldots, I_{mn+1}$ be closed intervals on the real line (i.e., $I_j = [a_j, b_j]$ with $a_j, b_j$ real numbers, for $j = 1, 2, \ldots, mn + 1$). Then either there are $m + 1$ intervals that are pair-wise disjoint, or there are $n + 1$ intervals with nonempty intersection. (You may assume, for simplicity, that the intervals are ordered so that $a_1 \leq a_2 \leq \ldots \leq a_{mn+1}$.)

We define an ordering by $I_i \leq I_j$ if $b_i < a_j$, i.e., if the intervals are disjoint, and we also define $I_i \leq I_i$ for all $i$. What this means then is that chains are sets of intervals which are mutually pair-wise disjoint, and antichains are composed of intervals with nonempty intersection. The rest follows from Dilworth’s theorem.