P1 Let $A_n$ be the graph obtained from $K_n$ by deleting an edge. How many spanning trees does $A_n$ have?

P2 How many spanning trees does the complete bipartite graph $K_{m,n}$ have?

P3 Let $G$ be a bipartite graph in which each vertex has degree exactly $d$. Show that there are at least $d$ different perfect matchings in $G$.

P4* An $n \times n$ matrix is called a permutation matrix if all its entries are equal to 0 or 1, and there is precisely one 1 in each row and in each column. (For example, the identity matrix $I_n$ is a permutation matrix.)

a) Given a permutation matrix $P$, consider the bipartite graph $G$ whose left class $X$ is indexed by the rows and the right class $Y$ is indexed by the columns of $P$, and for which $(x,y)$ with $x \in X$ and $y \in Y$ is an edge if and only if $P_{xy} = 1$.

What kind of bipartite graph is $G$?

b) Show that if $M$ is an $n \times n$ matrix with all entries equal to 0 or 1 and with exactly $m$ 1s in each row and exactly $m$ 1s in each column, then $M$ can be written as a sum of $m$ permutation matrices.

P5 There are $n$ applicants for some set of $m$ jobs. Assume that for any $2 \leq k \leq n$, every $k$ applicants have applied to at least $k - 2$ jobs. Show that it is possible to match $n - 2$ or more of applicants to jobs they applied for.

P6 (Supplementary Exercise 19, chapter 11) A graph is called color critical if it has chromatic number $k$, but if we delete any vertex of the graph together with its incident edges, we get a graph of chromatic number $k - 1$.

a) Give an example of a color-critical graph for $k = 3$. The example should not be $K_3$.

b) Given a non-complete graph example of a color-critical graph for $k = 4$. The example should not be $K_4$.

P7* Let $G$ be a graph on 11 vertices, and $G^c$ be its complement. Show that at least one of $G$ and $G^c$ is not planar.

P8

a) Suppose we delete two edges from $K_6$. Is the resulting graph planar? What about if we delete three edges?
b) A planar graph $G$ has 16 vertices and 40 edges. How many triangular regions are there in the planar drawing of $G$, if all the regions (including the unbounded one) are either triangles or quadrilaterals?

**P9** Consider a group of 8 people, each pair of which are either friends or enemies. Show that if some person in the group has at least 6 friends in the group, then either there are 4 people who are mutual friends or 3 people who are mutual enemies.

**P10** Suppose that $n$ people attend a party. In any group of 3 guests, there are two who do not like each other. In any group of 7 guests, there are two who do like each other. At the end of the party, each person gives a gift to all the people they like.

Show that at most $6n$ gifts have been given at the party.

**P11** Given the complete graph on $2n$ vertices $K_{2n}$, show that there exists a coloring of the edges on $K_{2n}$ with $n$ colors which has fewer than $4n/3$ monochromatic triangles.

**P12** Consider the graph $K_n$, and pick a uniformly random 2-coloring of its edges. How many monochromatic $K_4$s do we expect $G$ to have?

**P13** We say that a permutation $\pi$ of $[n]$ transposes $i$ and $j$ if $\pi(i) = j$ and $\pi(j) = i$. For instance, if $n = 7$ and $\pi = (5, 3, 2, 6, 1, 7, 4)$ then $\pi$ transposes 1 and 5, and it also transposes 2 and 3.

What is the expected number of transpositions in a random permutation of $[n]$? Assume all permutations are equally likely.

**P14** What is the expected number of leaves in a uniformly random tree with vertex set $[n]$, and what fraction of the vertices are expected to be leaves as $n \to \infty$?

**P15**

1) Let $P$ be the poset of numbers up to $n$ ordered by divisibility ($x \leq y$ if $x$ divides $y$). What is the size of the longest chain in $P$?

2) Recall $B_n$, the poset of the power set of $[n]$ (the set of all subsets of $[n]$) ordered by inclusion. What is the length of the longest chain in $B_n$?

**P16** Let $I_1, \ldots, I_{mn+1}$ be closed intervals on the real line (i.e., $I_j = [a_j, b_j]$ with $a_j, b_j$ real numbers, for $j = 1, 2, \ldots, mn + 1$). Then either there are $m + 1$ intervals that are pair-wise disjoint, or there are $n + 1$ intervals with nonempty intersection. (You may assume, for simplicity, that the intervals are ordered so that $a_1 \leq a_2 \leq \ldots \leq a_{mn+1}$.)