Instructions:

- *Any attempt at cheating will be dealt with severely.*
- No books or notebooks allowed; you may use an 8.5 × 11 double-sided, handwritten sheet of notes for personal use (do not share).
- Read problems carefully.
- Justify all your work.
- Raise your hand if you have a question.
- If you need more space, request additional blank sheets. It is your responsibility to have your instructor staple the additional sheets to your exam before you turn it in.
- Please turn off cell phones. GOOD LUCK!
Problem 1. (10pts) Prove that, if we 2-color the points in the plane with the colors red and blue, given any $d > 0$, there exists a segment of length $d$ with same-colored endpoints.

Solution. Choose three points in the plane that form an equilateral triangle of side $d$. By the Pigeonhole Principle, two of them have to have the same color. Those two are the same-color endpoints of a segment of length $d$. 
Problem 2. (15pts) Recall the notion of duality for planar graphs, and the fact that the dual of a simple graph is not necessarily simple.

(a) Compute the dual of a path on $k$ vertices.

(b) Compute the dual of a cycle on $k$ vertices.

(c) Compute the dual of the wheel graph on $k$ vertices (Figure 1).

Solutions.

(a) (5pts) The dual is single vertex with $k$ loops attached to it.

(b) (5pts) The dual is a graph with 2 vertices connected by $k$ edges.

(c) (5pts) The dual is itself.

Figure 1
Problem 3. (15pts) Recall the chromatic number of a graph.

(a) What is the chromatic number of a $k$-tree?

(b) What is the chromatic number of a cycle on $k$ vertices?

(c) What is the chromatic number of the wheel graph on $k$ vertices of Figure 1 (previous page)?

Solutions.

(a) (5pts) The chromatic number of a tree is 2 (since a tree has no cycles, it has no odd cycles).

(b) (5pts) If $k$ is even, the chromatic number is 2; if it is odd, the chromatic number is 3.

(c) (5pts) If $k$ is even, the long cycle in the wheel has an odd number of vertices, therefore one needs at least 3 color to color it; one extra color will be needed for the middle vertex, so the chromatic number is 4.

If $k$ is odd, the long cycle in the wheel is even, therefore 2-colorable; we need another color for the middle vertex, therefore the chromatic number is 3.
Problem 4. (21pts) Consider $H_n$ to be the number of $n$-permutations containing only cycles of length 2 and 3 (for example, for $n = 4$, $(12)(34)$ is in $H_4$, while $(1)(234)$ and $(1)(24)(3)$ are not).

(a) Find a recurrence for $H_n$.

(b) Let $H_0 = 1$, $H_1 = 0$, $H_2 = 1$. Let $H(x) = \sum_{i=0}^{\infty} H_i \frac{x^i}{i!}$ be the exponential generating function for $\{H_n\}_{n \in \mathbb{Z}_+}$. Find $H(x)$. (Hint: recall that $\frac{H'(x)}{H(x)} = (\ln H(x))'$.)

(c) Find a closed-form, sum expression for $H_n$ (either by using the exponential generating function found in (b), or by using the formula for the number of permutations with prescribed cycle structure).

Solutions.

(a) (7pts) If $n$ is in a 2-cycle, there are $(n-1)$ ways to pick the other number in the 2-cycle. If $n$ is in a 3-cycle, there are $n^2$ ways to pick the other 2 numbers, and for each triplet, there are 2 ways to form the 3-cycle. This yields

$$H_n = (n-1)H_{n-2} + (n-1)(n-2)H_{n-3}.$$ 

(b) (7pts) We multiply the above by $\frac{x^{n-1}}{(n-1)!}$ and sum it over all $n \geq 3$, to obtain

$$\sum_{n=3}^{\infty} H_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=3}^{\infty} H_{n-2} \frac{x^{n-1}}{(n-2)!} + \sum_{n=3}^{\infty} H_{n-3} \frac{x^{n-1}}{(n-3)!}.$$

$$H'(x) - H_2x - H_1 = x(H(x) - H_0) + x^2H(x)$$

$$H'(x) = H_2x + H_1 - H_0x + (x + x^2)H(x).$$

Since $H_2 = 1 = H_0$ and $H_1 = 0$, the above becomes

$$H'(x) = (x + x^2)H(x),$$

$$\frac{H'(x)}{H(x)} = (x + x^2),$$

$$(\ln H(x))' = x + x^2,$$

$$\ln H(x) = \frac{x^2}{2} + \frac{x^3}{3} + C,$$

$$H(x) = Ce^{\frac{x^2}{2} + \frac{x^3}{3}},$$

and since $H(0) = H_0 = 1$, it follows that $C = 1$, i.e. $H(x) = e^{\frac{x^2}{2} + \frac{x^3}{3}}$.

(c) (7pts) Either by writing $H(x) = e^{x^2/2}e^{x^3/3} = \sum_{i=0}^{\infty} \frac{x^i}{2^i i!} \sum_{j=0}^{\infty} \frac{x^j}{3^j j!}$ or by using the fact that the number of $n$-permutations with $i$ cycles of length 2 and $j$ cycles of length 3 is $\frac{n!}{2^i 3^j i! j!}$, we get

$$H_n = \sum_{2i + 3j = n} \frac{n!}{2^i 3^j i! j!}.$$
Problem 5. (13pts)

(a) How many spanning trees does the graph in Figure 2 have?
(b) How many spanning trees does the graph in Figure 3 have?

Solutions.

(a) (8pts) We have to “destroy” the cycle (14, 15, 16), and exactly 2 of the 3 paths between 1 and 14. The cycle has length 3 and each of the paths have length 5. There are \( \binom{3}{2} \) ways to choose 2 of the 3 paths and once they’re chosen, there are \( 5^2 \) ways to break them. So the total number of ways to do obtain a minimally connected graph is

\[
3 \cdot \binom{3}{2} \cdot 5^2 = 9 \cdot 25 = 225.
\]

(b) (5pts) We will disregard the cycle (14, 15, 16); it will just give us 3 times as many options at the end. We have to break the cycle (9, 13, 14), which we can do as follows:

1. We can eliminate all three edges. Then the resulting graph is a tree. (1).
2. We can eliminate precisely 2 edges.
   - (9, 14) and (13, 14). We keep (9, 13). Then we need to break the cycle (1→9→13→1), but keep (9, 13); we can do that in (8) ways.
   - (9, 14) and (9, 13). We must eliminate the cycle (1→14→13→1), but keep the edge (14, 13). We can do that in (9) ways.
   - (13, 14) and (9, 13). Similar with the one above: (9) ways.
3. We can eliminate only 1 edge. Regardless of which edge we eliminate, we must destroy 2 of the 3 paths (1→5→14), (1→6→9), and (1→10→13). We can do that in \( 5 \cdot 4 + 4 \cdot 4 + 5 \cdot 4 = 56 \) ways; since there are three ways to choose an edge to eliminate, we get a total of \( 56 \cdot 3 = 168 \) ways.

So the total number of spanning trees is \( 3 \cdot (1 + 8 + 9 + 168) = 585 \).
Problem 6. (13pts) Fix an \( n \)-permutation \( p \) with \( k \) cycles.

(a) Choose one of the \( k \) cycles at random. What is its average length \( l_{n,k} \)?

(b) Assume now that \( l_{n,k} \) is an integer, and let \( p \) be a random \( n \)-permutation containing \( k \) cycles. What is the probability that \( p \) contains a given cycle of length \( l_{n,k} \)? You may express the answer in terms of a well-known type of number.

Solutions.

(a) (8pts) Let \( l_1, \ldots, l_k \) be the lengths of the cycles. Then the expected value is

\[
E(l_{n,k}) = \frac{1}{k}(l_1 + l_2 + \ldots + l_k).
\]

But \( l_1 + l_2 + \ldots + l_k = n \), thus \( l_{n,k} = \frac{n}{k} \).

(b) (5pts) There are \( c(n, k) \) permutations with \( k \) cycles. If we require that one of these cycles is given and of length \( l_{n,k} \), the other \( k-1 \) cycles can be formed with the remaining \( (n - l_{n,k}) = (n - \frac{n}{k}) \) elements. Therefore the probability is

\[
p = \frac{c(n - \frac{n}{k}, k - 1)}{c(n, k)}.
\]
Problem 7. (13pts) Assume that $f$ and $g$ are functions from the positive integers $\mathbb{Z}_+$ to the positive real numbers $\mathbb{R}_+$, such that $f(n) = O(g(n))$. (Note: the constant in the definition of Big-Oh is positive.)

(a) Is $(f(n))^2 = O((g(n))^2)$?

(b) Find a pair $(f, g)$ such that $|\log_2(f(n))| \neq O(|\log_2(g(n))|)$.

Solutions.

(a) (8pts) Yes:

$$0 < f(n) \leq cg(n) \iff f^2(n) \leq c^2g^2(n).$$

(b) (5pts) An example is provided by $f(n) = 2^{-n}$, $g(n) = 1$: although $f(n) \leq g(n)$, $|\log_2(f(n))| = |-n| = n$, while $|\log_2(g(n))| = 0$. 