P1.

1. Prove that if \( x \) is a non-zero rational number and \( y \) is an irrational number, then the product \( xy \) is not rational.

2. Does it follow that if the product \( xy \) is rational, then both \( x \) and \( y \) must be rational? Prove or give a counterexample.

P2. Read the proof for the fact that \( 0.\bar{9} = 1 \) from the textbook (Example 13.3.3). Adapt it to show that if a non-zero rational number \( x \) has a finite decimal representation

\[
x = a_0.a_1a_2\ldots a_n,
\]

where \( a_n > 0 \) if \( n > 0 \), then \( x \) can also be represented as

\[
x = a_0.a_1a_2\ldots(a_n - 1)\bar{9}
\]

P3. (You will have to wait on this until Monday). Use Cantor’s diagonal argument to show that the interval of real numbers \([1, 2]\) is uncountable.

P4. Prove that if \( A \) and \( B \) are enumerable sets, then \( A \cup B \) is also enumerable. You may assume that they are disjoint. (Hint: consider the Cantor-Hilbert Hotel argument, let \( f : \mathbb{Z}^+ \to A \) and \( g : \mathbb{Z}^+ \to B \) be the bijections from \( \mathbb{Z}^+ \) to \( A \), respectively \( B \), and construct a bijection \( h : \mathbb{Z}^+ \to A \cup B \) from \( f \) and \( g \) in a way mimicking the “freeing up all odd numbers, then using them to lodge the enumerable set of newcomers” argument.)

P5. (You will have to wait on this until Monday, and you may assume the Continuum Hypothesis.)

Determine the cardinalities for each of the sets below. Your answer should be a number or a cardinality, like \( \aleph_0 \), \( \aleph_1 \), \( \mathcal{P}(\mathbb{R}) \) etc. Give a brief justification.

1. the irrational numbers
2. \( \mathbb{Q} \times \mathbb{Q} \)
3. \( S = \{ \sqrt{n} \mid n \in \mathbb{Q} \text{ and } n \geq 0 \} \)
4. \( A = \{ n \in \mathbb{Z} \mid 15 \leq n^2 - 1 \leq 35 \} \)
5. the power set of the reals, \( \mathcal{P}(\mathbb{R}) \)

P6. (Wait on this until Monday.) For a subset \( A \subseteq \mathbb{R} \), let \( \chi_A : \mathbb{R} \to \mathbb{R} \) be the characteristic function of \( A \), defined by

\[
\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A, \\
0, & \text{if } x \in A^c.
\end{cases}
\]
Let $S$ be the set of all characteristic functions, that is,

$$S = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a characteristic function for some } A \subseteq \mathbb{R} \}.$$ 

What is the cardinality of $\mathcal{P}(S)$?

**P7.** Give examples of sets (other than precisely $\mathbb{N}_{81}$, $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) of the following cardinalities:

1. 81
2. 0
3. $\aleph_0$
4. $\aleph_1$
5. $\beth_2$