Homework consists of four parts: the written assignment W1-W4 which counts towards the final grade for the course, the optional bonus problems that are recommended for those who are interested in taking the Putnam competition, the presentation assignment (which will not be handed in) and (occasionally) a reading assignment.

Please don’t get discouraged if you cannot immediately solve all of the problems, especially the presentation problems and the bonus problems. The instructors are more than happy to discuss any of the problems by e-mail and in person - in particular, during Monday office hours - before the assignment is due. Hints are given upon request.

This homework has no reading assignment.

Written assignment (4 problems).

W1. Let $p$ and $q$ be prime numbers. Show that $\sqrt{p} + \sqrt{q}$ is an irrational number.

W2. A chessboard of size $2^n \times 2^n$ has a $1 \times 1$ square in the upper left corner cut out from it. For which positive integers $n$ is it possible to cut the remaining board into the “ell” (or “tromino”) shapes?

W3. You are given 2012 numbers. Show that you can choose several numbers such that their sum is divisible by 2012.

W4. The sequence of Fibonacci numbers is defined by

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad \text{if } n \geq 2.$$ 

(a) Prove that any positive integer can be represented as a sum of several different Fibonacci numbers.

(b) Prove that $F_n$ is divisible by 3 if and only if $n$ is divisible by 4.

Bonus problem.

B1. Define

$$q(n) = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor$$

Determine (with proof) all positive integers $n$ such that

(i) $q(n) < q(n + 1), \quad \text{and} \quad n \neq 2012, 2013$.
(ii) $q(n) = q(n + 1)$,

(iii) $q(n) > q(n + 1)$.

Presentation assignment (4 problems).

**P1.** Twenty one adventurous students participated in a donut eating contest. The total number of donuts consumed was 200. Prove that there were two participants who ate the same number of donuts.

**P2.** Prove that any positive integer can be written as a sum of several (one or more) numbers of the form $2^n3^m$, for $n, m \geq 0$ and such that no summand divides another.

**P3.**

(a) What is the largest number of squares on an $8 \times 8$ chessboard which can be colored purple, so that any ell (aka tromino, see Figure 1) on the board has at least one square which is not colored purple?

(b) What is the smallest number of squares on an $8 \times 8$ chessboard which can be colored purple, so that any tromino on the board has at least one square which is colored purple?

**P4.** Prove that there exists an integer whose decimal representation consists only of 1’s, and which is divisible by 2011.